

# RATIONALLY SIMPLY CONNECTED VARIETIES AND PSEUDO ALGEBRAICALLY CLOSED FIELDS

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**ABSTRACT.** The cohomological dimension of a field is the largest degree with non-vanishing Galois cohomology. Serre’s “Conjecture II” predicts that for every perfect field of cohomological dimension 2, every torsor over the field for a semisimple, simply connected algebraic group is trivial. A field is perfect and “pseudo algebraically closed” (PAC) if every geometrically irreducible curve over the field has a rational point. These have cohomological dimension 1. Every transcendence degree 1 extension of such a field has cohomological degree 2. We prove Serre’s “Conjecture II” for such fields of cohomological degree 2 provided either the field is of characteristic 0 or the field contains primitive roots of unity for all orders  $n$  prime to the characteristic. The method uses “rational simple connectedness” in an essential way. With the same method, we prove that such fields are  $C_2$ -fields, and we prove that “Period equals Index” for the Brauer groups of such fields. Finally, we use a similar method to reprove and extend a theorem of Fried-Jarden: every perfect PAC field of positive characteristic is  $C_2$ .

## 1. STATEMENT OF RESULTS

For a field  $L$ , the *cohomological dimension* is the supremum (possibly infinite) over all integers  $n$  such that there exists a discrete Galois module with non-vanishing degree  $n$  Galois cohomology. For every finite extension  $L'/L$ , the cohomological dimension of  $L'$  is no greater than the cohomological dimension of  $L$ , [Ser02, Proposition II.10, p. 83]. The cohomological dimension equals 0 if and only if the field is separably closed. A *Severi-Brauer variety* of dimension  $n - 1$  over  $L$  is a smooth, projective  $L$ -scheme  $X$  such that  $X \times_{\mathrm{Spec} L} \mathrm{Spec} (L^{\mathrm{sep}})$  is isomorphic to  $\mathbb{P}_{L^{\mathrm{sep}}}^{n-1}$ . These are in bijection with the torsors over  $L$  for the semisimple adjoint group  $\mathbf{PGL}_n = \mathrm{Aut}(\mathbb{P}^{n-1})$ , which is connected but not simply connected, via the  $L$ -scheme of isomorphisms between  $X$  and  $\mathbb{P}_L^{n-1}$ . The *period*, or *exponent*, of  $X$  equals the smallest integer  $d > 0$  such that there exists an invertible sheaf on  $X$  whose base change to  $X \otimes_L L^{\mathrm{sep}} \cong \mathbb{P}_{L^{\mathrm{sep}}}^{n-1}$  is isomorphic to  $\mathcal{O}_{\mathbb{P}_{L^{\mathrm{sep}}}^{n-1}}(d)$ . The *index* equals the smallest integer  $m$  such that there exists a closed subscheme  $Y$  of  $X$  whose base change in  $\mathbb{P}_{L^{\mathrm{sep}}}^{n-1}$  is a linear subvariety of dimension  $m - 1$ . The period divides the index, the index divides  $n$ , and the period and index have the same prime factors [GS06, Proposition 4.5.13]. The *Period-Index Problem* asks for the smallest integer  $e$  (assuming one exists), such that always the index divides the period raised to the power  $e$ .

For a perfect field  $L$ , the cohomological dimension of  $L$  is  $\leq 1$  if and only if for every finite separable extension  $L'/L$ , every Severi-Brauer variety over  $L'$  has period

1, i.e., every  $\mathbf{PGL}_n$ -torsor over  $L'$  is trivial [Ser02, Proposition II.5, p. 78]. For imperfect fields, typically this condition is taken as the definition of *dimension*  $\leq 1$ , as opposed to “cohomological dimension  $\leq 1$ ”. Serre formulated a strong converse, “Conjecture I”: for every field of dimension  $\leq 1$ , every torsor over  $L$  for every semisimple and connected algebraic group is trivial. This was proved by Steinberg, [Ste65]. For a perfect field  $L$ , by a theorem of Merkurjev-Suslin [Sus84, Corollary 24.9], the cohomological dimension is  $\leq 2$  if and only if all  $\mathbf{SL}_D$ -torsors over  $L$  are trivial for those semisimple and *simply connected* algebraic groups  $\mathbf{SL}_D$  arising as inner forms of  $\mathbf{SL}_n$  over  $L$ . Serre formulated a strong converse, “Conjecture II”: for every perfect field of cohomological dimension  $\leq 2$ , every torsor over  $L$  for every semisimple and simply connected algebraic group is trivial. Serre also formulated a version of his conjecture for imperfect fields of characteristic  $p$  [Ser95, Section 5.5]: Serre adds the hypotheses that  $[L : L^p] \leq p^2$  and that  $H_p^3(F')$  is zero for all finite, separable extensions  $F'/F$ .

A field  $L$  is *perfect*, resp. *perfect and pseudo-algebraically closed* (PAC), if every quasi-projective  $L$ -scheme that is geometrically irreducible and zero-dimensional, resp. one-dimensional, has an  $L$ -point. Since every perfect PAC field  $L$  is infinite, Bertini theorems imply that every quasi-projective  $L$ -scheme  $X$  that is geometrically irreducible contains a closed subscheme that is geometrically irreducible of dimension 1 (or dimension 0 if  $X$  has dimension 0). Thus, for a perfect PAC field, every quasi-projective  $L$ -scheme  $X$  that is geometrically irreducible has an  $L$ -point. Via Weil’s restriction of scalars, every finite extension of a perfect PAC field is again a perfect PAC field. Since Severi-Brauer varieties are geometrically irreducible, every Severi-Brauer variety over a perfect PAC field has a rational point, and thus it has index 1 (so also it has period 1). Therefore, every perfect PAC field has dimension  $\leq 1$ . Thus, every function field  $K/L$  of transcendence degree 1 over a perfect PAC field  $L$  has cohomological dimension  $\leq 2$ , [Ser02, Proposition II.11, p. 83]. If  $L$  has characteristic  $p$ , then  $K$  is imperfect. Nonetheless,  $[K : K^p]$  equals  $p$ , so Serre’s modified version of “Conjecture II” predicts triviality for all  $G$ -torsors over  $K$  for semisimple and simply connected algebraic groups  $G$ . A perfect PAC field is *nice* if either it has characteristic 0 or if it has positive characteristic  $p$  and it contains a primitive root of unity of order  $n$  for every integer  $n$  prime to  $p$ . Jarden and Pop proved the following theorem under the hypothesis that the field has characteristic 0 *and* contains all roots of unity, [JP09].

**Theorem 1.1.** *For every perfect PAC field  $L$  that either has characteristic zero or contains a primitive root of unity of order  $n$  for every integer  $n$  prime to the characteristic  $p$ , for every function field  $K/L$  of transcendence degree 1, every torsor over  $K$  for every semisimple and simply connected algebraic group is trivial.*

The same method of proof also proves Period equals Index for  $K$ . We thank Max Lieblich who shared with us his independent (and different) proof of the following theorem. This theorem can also be proved using the Hasse principle of Efrat, [Efr01].

**Theorem 1.2.** *For every perfect PAC field  $L$  that is nice, for every function field  $K/L$  of transcendence degree 1, every Severi-Brauer variety over  $K$  has period equal to index.*

A smooth, projective, geometrically connected scheme  $X$  over a field  $K$  is *Fano*, resp. *2-Fano*, etc., if the first graded piece of the Chern character of  $T_{X/K} = (\Omega_{X/K})^\vee$  is positive, resp. if the first two graded pieces are positive, etc., cf. [dJS07]. A field  $K$  is  $C_1$ , resp.  $C_2$ , etc., if every  $K$ -scheme in  $\mathbb{P}_K^{n-1}$  that is a specialization of Fano complete intersections, resp. 2-Fano complete intersections, has a  $K$ -point. The method proves that every function field  $K$  of transcendence degree 1 over a nice, perfect PAC field  $L$  is a  $C_2$ -field, first proved by Fried-Jarden, [FJ05, Theorem 21.3.6]. In fact, there are by now many examples of rationally simply connected varieties beyond 2-Fano complete intersections. The following formulation includes one such family discovered by Robert Findley, [Fin10].

**Theorem 1.3.** *Let  $L$  be a PAC field that is nice, and let  $K/L$  be a function field of transcendence degree 1. Let  $X$  be a  $K$ -scheme, and let  $\mathcal{L}$  be an invertible sheaf on  $X$ . Then  $X$  has a  $K$ -point in either of the following cases: first,  $X \otimes_K K^{sep}$  is isomorphic to the common zero locus in  $\mathbb{P}_{K^{sep}}^{n-1}$  of  $c$  homogeneous polynomials of degrees  $(d_1, \dots, d_c)$  such that  $d_1^2 + \dots + d_c^2 < n$ , and the base change of  $\mathcal{L}$  is the restriction of  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ . Second, there is a  $K$ -point if  $X \otimes_K K^{sep}$  is isomorphic to the intersection of a degree  $d$  hypersurface and  $\text{Grass}_{K^{sep}}(r, (K^{sep})^{\oplus n})$ , with its Plücker embedding, such that  $(3r-1)d^2 - d < n - 4r - 1$ , and the base change of  $\mathcal{L}$  is the restriction of the Plücker  $\mathcal{O}(1)$ .*

Now let  $(R, \mathfrak{m}_R)$  be a Henselian DVR whose fraction field  $K$  has characteristic 0 and whose residue field  $L$  is a perfect PAC field of characteristic  $p$  that contains a primitive root of unity of order  $n$  for every integer  $n$  prime to  $p$ . The Ax-Kochen method of Denef, [Den16], yields the following.

**Theorem 1.4.** *The field  $K$  has cohomological dimension  $\leq 2$ . For Severi-Brauer varieties over  $K$ , the Period equals the Index. There exists an integer  $p_0$  such that for every such field with characteristic  $p \geq p_0$ , Serre’s “Conjecture II” holds for  $K$ . For every integer  $n$  and sequence of integers  $(d_1, \dots, d_c)$  with  $d_1^2 + \dots + d_c^2 \leq n-1$ , there exists an integer  $p_0 = p_0(n; d_1, \dots, d_r)$  such that for every field  $K$  as above of characteristic  $p \geq p_0$ , every closed subscheme of  $\mathbb{P}_K^{n-1}$  defined by equations of degrees  $(d_1, \dots, d_c)$  has a  $K$ -point. Finally, for all triples of integers  $(n, r, d)$  with  $(3r-1)d^2 - d < n - 4r - 1$ , there exists an integer  $p_0 = p_0(n, r, d)$  such that for every  $K$  as above of characteristic  $p \geq p_0$ , for every polarized  $K$ -scheme  $(X_K, \mathcal{L}_K)$  whose base change to  $\overline{K}$  is a degree  $d$  hypersurface in  $\text{Grass}_{\overline{K}}(r, \overline{K}^{\oplus n})$  with its Plücker invertible sheaf,  $X_K$  has a  $K$ -point.*

Finally, a variant of the method implies a similar result for perfect PAC fields of characteristic  $p$  that are not necessarily nice. The  $C_2$  result was first proved by Fried-Jarden, [FJ05, Theorem 21.3.6], but there are many other rationally simply connected varieties than 2-Fano complete intersections.

**Theorem 1.5.** *Let  $L$  be a perfect PAC field that is not necessarily nice. Let  $X$  be an  $L$ -scheme, and let  $\mathcal{L}$  be an invertible sheaf on  $X$ . Then  $X$  has a  $L$ -point in either of the following cases: first,  $X \otimes_L \overline{L}$  together with  $\mathcal{L}$  is isomorphic to the common zero locus in  $\mathbb{P}_{\overline{L}}^{n-1}$  of  $c$  homogeneous polynomials of degrees  $(d_1, \dots, d_c)$  such that  $d_1^2 + \dots + d_c^2 < n$  together with the restriction of  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ . Second, there is a  $K$ -point if  $X \otimes_L \overline{L}$  is isomorphic to the intersection of a degree  $d$  hypersurface*

nd  $\text{Grass}_{\overline{L}}(r, \overline{L}^{\oplus n})$ , with its Plücker embedding, such that  $(3r-1)d^2 - d < n-4r-1$ , and the base change of  $\mathcal{L}$  is the restriction of the Plücker  $\mathcal{O}(1)$ .

The method uses very much the notions of *rationally connected* and *rationally simply connected* varieties, together with their specializations. The full results are considerably stronger than the formulations above: they also include results about arbitrary specializations, and they give height bounds for rational points.

**Acknowledgments.** I am very grateful to Chenyang Xu; this article builds on the earlier joint work in [SX]. I am very grateful to Max Lieblich who explained his independent proof that “Period equals Index” for function fields over PAC fields. I am grateful to Yi Zhu with whom I discussed the technique to transport from characteristic 0 to characteristic  $p$  results around Serre’s “Conjecture II” using Nisnevich’s solution of the Grothendieck-Serre conjecture in dimension 1. During the development of this article I was supported by NSF Grants DMS-0846972 and DMS-1405709, as well as a Simons Foundation Fellowship.

## 2. A BERTINI THEOREM OVER NON-ALGEBRAICALLY CLOSED FIELDS

The following Bertini theorem extends [GS13, Corollary 2.2] to arbitrary fields. Let  $k$  be a field. Let  $B$  be a separated, geometrically integral  $k$ -scheme of dimension  $m \geq 1$ . Up to replacing  $B$  by a dense, open subscheme, assume that  $B$  is a normal scheme. Let  $u : B \rightarrow \mathbb{P}_k^N$  be a generically unramified, finite type morphism. Let  $h : B' \rightarrow B$  be a generically finite, finite type morphism. Let  $c$  be an integer satisfying  $0 \leq c \leq \max(0, m-1)$ , and let  $r$  denote  $N - c$ . Denote by  $\text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)$  the Grassmannian parameterizing linear subspaces of  $\mathbb{P}_k^N$  of dimension  $r$ , i.e., of codimension  $c$ . This is the Hilbert scheme of  $\mathbb{P}_k^N$  over  $k$  for the numerical polynomial  $P_r(t)$  such that  $P_r(d) = \binom{d+r}{r}$  for every integer  $d \geq -r$ . Denote by  $\Lambda \subset \text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N) \times_{\text{Spec } k} \mathbb{P}_k^N$  the universal linear subspace. For every field extension  $K/k$ , for every  $[L] \in \text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)(\text{Spec } K)$ , the associated morphism

$$h_L : B' \times_{\mathbb{P}_k^N} L \rightarrow B \times_{\mathbb{P}_k^N} L,$$

is a morphism of finite type  $K$ -schemes.

**Definition 2.1.** An *inseparable section* of  $h$  is an integral closed subscheme  $Z \subset B'$  such that  $h|_Z : Z \rightarrow B$  is dominant and the field extension  $k(Z)/k(B)$  is purely inseparable. A *rational section* is an inseparable section such that the extension  $k(Z)/k(B)$  is an isomorphism of fields. The *domain of definition* is the maximal open subscheme of  $B$  over which  $h|_Z$  is flat and finite. Equivalently, this is the maximal open subscheme over which  $h|_Z$  is faithfully flat.

For every inseparable section, resp. rational section,  $Z$  with domain of definition  $W$ , if  $B \times_{\mathbb{P}_k^N} L$  intersects  $W \times_{\text{Spec } k} \text{Spec } K$  in a dense open subset of  $B \times_{\mathbb{P}_k^N} L$ , then the associated reduced scheme of  $Z \times_{\mathbb{P}_k^N} L$  is an inseparable section, resp. rational section, of  $h_L$ .

**Theorem 2.2.** *There exists a dense Zariski open subset  $U \subset \text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)$  and a finite, Galois extension  $k'/k$  such that for every field extension  $K/k$  that is linearly disjoint from  $k'$ , for every  $[L] \in U(\text{Spec } K)$ , the restriction map from the set of inseparable sections of  $h$  to the set of inseparable sections of  $h_L$  is well-defined and*

is a bijection. If  $k$  is perfect, the same holds with “rational sections” in place of “inseparable sections”.

*Proof.* If  $c$  equals 0, then  $r$  equals  $N$ , and the result is vacuously true with  $k' = k$ . Thus, assume that  $m \geq 2$ , and assume that  $1 \leq c \leq m - 1$ .

**Restriction Map Well-Defined and Injective.** By [Jou83, Théorème 4.10, 6.10], there exists a dense open subscheme  $U_0 \subset \text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)$  such that for every  $K/k$  and for every  $L \subset \mathbb{P}_K^N$  with  $[L] \in U_0(K)$ ,  $B \times_{\mathbb{P}_k^N} L$  is a geometrically integral  $K$ -scheme.

There are only finitely many inseparable sections of  $h$ , resp. rational sections of  $h$ . For each, the maximal domain of definition is a dense open subset whose complement is a proper closed subset. Since  $u$  is generically finite, the image of this proper closed subset in  $\mathbb{P}_k^N$  is contained in a proper closed subset. Similarly, for any two distinct inseparable sections, resp. rational sections, the intersection of the closures of the images is a closed subset of  $X$  whose closed image in  $B$  does not contain the generic point. Thus, the image of this closed subset in  $\mathbb{P}_k^N$  is contained in a proper closed subset. Since  $\mathbb{P}_k^N$  is irreducible, the union of finitely many proper closed subsets is a proper closed subset, say  $C$ . The Fano scheme parameterizing linear subspaces contained in  $C$  is a proper closed subset of  $\text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)$ . Replace  $U_0$  by the relative complement in  $U_0$  of this Fano scheme. Then for every  $K/k$ , for every  $L \subset \mathbb{P}_K^N$  with  $[L] \in U_0(\text{Spec } K)$ ,  $B \times_{\mathbb{P}_k^N} L$  is a geometrically integral  $K$ -scheme that intersects the domain of definition of each inseparable section, resp. rational section, of  $h$  and such that for any two inseparable sections, resp. rational sections,  $B \times_{\mathbb{P}_k^N} L$  is not contained in the closure of the locus over which the two sections are equal. Thus, the base change over  $K$  of every inseparable section, resp. rational section, of  $h$  restricts to a well-defined inseparable section, resp. rational section, of  $h_L$ , and this restriction map is injective.

**Surjectivity of Restriction Map. Noetherian Induction.** It remains to prove that there exists a dense open subset  $U$  of  $U_0$  and a finite Galois extension  $k'/k$  such that for every finite  $K/k$  that is linearly disjoint from  $k'$ , for every  $L \subset \mathbb{P}_K^N$  with  $[L] \in U(\text{Spec } K)$ , the restriction map is surjective. This is proved by Noetherian induction for restrictions of  $h$  to closed subsets of  $B'$ . If  $B'$  equals  $C \cup Y$  for proper closed subsets  $C$  and  $Y$ , and if the result is proved for  $C$  and  $Y$ , then define  $k'/k$  to be the compositum of  $k'_C/k$  and  $k'_Y/k$ , and define  $U = U_C \cap U_Y$ . Since  $B$  is integral, resp. since  $B \times_{\mathbb{P}_k^N} L$  is integral, every inseparable section, resp. rational section, has image in  $C$  or in  $Y$ . Thus, the results for  $C$  and  $Y$  imply the result for  $B'$ . Thus, assume that  $B'$  is irreducible.

Similarly, if  $B'$  is nonreduced, every inseparable section, resp. rational section, over  $B$ , resp. over  $B \times_{\mathbb{P}_k^N} L$ , factors through the associated reduced scheme of  $B'$ . Thus, assume that  $B'$  is irreducible and reduced.

**Case I. Morphism not Dominant.** If  $h(B')$  is contained in a proper closed subset of  $B$ , then the same argument as above shows that, for  $d = 1$  and for  $U$  a dense open subset of  $U_0$ , no  $B \times_{\mathbb{P}_k^N} L$  is contained in  $h(B')$ . Thus, the restriction map on sections is the unique set map from the empty set to the empty set, and the result is proved.

**Case II. Morphism Birational.** Similarly, if  $h : B' \rightarrow B$  is birational, then  $h_L$  is also birational since the domain of definition of the inverse rational section intersects the integral scheme  $B \times_{\mathbb{P}_k^N} L$ . Thus, the set of sections for each is a singleton set, and the restriction map is a bijection.

**Case III. Morphism Purely Inseparable, not Birational.** If  $h : B' \rightarrow B$  is dominant and purely inseparable of degree  $a > 1$ , then  $B'$  is an inseparable section. So again, the restriction map on inseparable sections is a bijection between singleton sets.

For rational sections, assume that  $k$  is perfect (otherwise the argument is much more technical). Since  $B'$  is integral and  $k$  is perfect,  $B'$  is generically smooth over  $k$ . Up to shrinking  $B$  and  $B'$ , assume that  $B$  and  $B'$  are  $k$ -smooth, and assume that  $h$  is finite and flat. Then  $dh^\dagger : h^*\Omega_{B/k} \rightarrow \Omega_{B'/k}$  is a homomorphism of locally free sheaves of rank  $m$ , and it is not surjective. Up to shrinking further, assume that the cokernel is locally free, so that also the image of  $dh^\dagger$  is locally free. Thus, also the kernel  $\mathcal{T}_g^\vee$  of  $dg^\dagger$  is locally free of positive rank  $e \geq 1$ . The fiber product

$$\Lambda'_{U_0} = U_0 \times_{\text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)} \Lambda \times_{\mathbb{P}_k^N} B',$$

parameterizes pairs  $([L], x)$  of a linear space  $L$  and a point  $x \in B'$  such that  $u(h(x)) \in L$ . By generic flatness, up to replacing  $U_0$  by a dense open subscheme, the projection morphism  $\Lambda_{U_0, X} \rightarrow U_0$  is flat.

By [Jou83, Théorème 4.10, 6.10], there is a dense open subset  $V_0 \subset \Lambda'_{U_0}$  parameterizing pairs  $([L], x)$  such that  $B \times_{\mathbb{P}_k^N} L$  is smooth of dimension  $m - c$  at  $h(x)$ . For every such  $([L], x)$ , the tangent space to  $B \times_{\mathbb{P}_k^N} L$  at  $h(x)$  gives a point in the Grassmannian bundle of  $(m - c)$ -dimensional subspaces of the Zariski tangent space  $T_{g(x)}B$ . The associated morphism from  $V_0$  to the Grassmannian bundle over  $B$  of the tangent bundle is dominant. Thus, there exists a dense open  $V \subset V_0$  parameterizing  $([L], x)$  such that the tangent space to  $B \times_{\mathbb{P}_k^N} L$  at  $h(x)$  is not contained in the annihilator of  $\mathcal{T}_g^\vee$  (this annihilator is a subspace of the Zariski tangent space of codimension  $\geq 1$ ). Since  $\Lambda'_{U_0} \rightarrow U_0$  is flat, the image in  $U_0$  of  $V$  is a dense Zariski open subscheme  $U \subset U_0$ . Set  $d$  equal to 1. For every field extension  $K/k$  and for every  $[L] \in U(K)$ , for the generic point of  $B \times_{\mathbb{P}_k^N} L$ , the derivative of  $h_L$  is not surjective. Therefore  $h_L$  admits no rational section.

**Case IV. Morphism Separable, not Birational.** Finally, assume that  $B' \rightarrow B$  is dominant and the separable closure  $L$  of  $k(B)$  in  $k(B')$  has degree  $a > 1$ . Denote by  $B''$  the integral closure of  $B$  in  $L$ . There is a factorization  $B' \rightarrow B'' \rightarrow B$ , and  $B'' \rightarrow B$  is dominant and generically étale of degree  $> 1$ . Up to shrinking  $B$  and  $B''$ , assume that  $B$  is regular, and assume that  $B'' \rightarrow B$  is finite and étale of degree  $a$ . Thus, there is no inseparable section nor rational section of  $B''/B$ . The goal is to find  $k'/k$  and  $U$  such that for every finite extension  $K/k$  that is linearly disjoint from  $k'/k$  and for every  $[L] \in U(\text{Spec } K)$ , also the finite étale morphism  $B'' \times_{\mathbb{P}_k^N} L \rightarrow B \times_{\mathbb{P}_k^N} L$  has no rational section (and thus no inseparable section). Then the restriction map is again the unique map between empty sets, which is a bijection. Up to replacing  $B'$  by  $B''$ , assume that  $h : B' \rightarrow B$  is finite and étale of degree  $a > 1$ .

**Case IVa. Base Field not Separably Closed in Extension.**  $[k' : k] > 1$ . Since  $B$  is generically smooth over  $k$  (being geometrically integral),  $k(B)/k$  is a

separable field extension. Thus also  $k(B')/k$  is a separable field extension. It is also a finitely generated field extension. Thus the algebraic closure of  $k$  in  $k(B')$  is a finite, separable extension  $\kappa/k$ . Denote by  $k'/k$  the Galois closure of  $\kappa/k$ . Note that  $[\kappa : k]$  is greater than 1 if and only if  $[k' : k]$  is greater than 1. In this case, set  $U$  equal to  $U_0$ . For every finite field extension  $K/k$  that is linearly disjoint from  $k'/k$ , and thus also linearly disjoint from  $\kappa/k$ , the composite morphisms

$$B' \times_{\mathbb{P}_k^N} L \rightarrow B \times_{\mathbb{P}_k^N} L \rightarrow L \rightarrow \text{Spec } K,$$

and

$$B' \times_{\mathbb{P}_k^N} L \rightarrow B' \rightarrow \text{Spec } \kappa,$$

establish that, as a  $K$ -scheme,  $B' \times_{\mathbb{P}_k^N} L$  factors through the nontrivial field extension  $K \otimes_k \kappa/K$ . Finally,  $K$  is algebraically closed in  $K(B \times_{\mathbb{P}_k^N} L)$ , since  $B \times_{\mathbb{P}_k^N} L$  is geometrically integral over  $K$ . Thus, there is no rational section of  $h_L$  (and thus there is no inseparable section).

**Case IVb. Base Field Separably Closed in Extension.**  $k' = k$ . In the final case, assume that  $k$  is already algebraically closed in  $X$ . Set  $k'$  equal to  $k$ . Now repeat the proof of [GS13, Corollary 2.2]. The composite morphism  $u \circ h : B' \rightarrow \mathbb{P}_k^N$  is generically unramified. Thus, repeating the argument above, there exists a dense open subset  $U \subset U_0$  such that for every  $K/k$  and every  $[L] \in U(K)$ ,  $B' \times_{\mathbb{P}_k^N} L$  is geometrically integral over  $K$ . Finally,  $h_L$  is a finite, flat morphism of degree  $a > 1$  between geometrically integral  $K$ -schemes. Thus, there is no rational section. This completes the proof by Noetherian induction.  $\square$

**Definition 2.3.** As above, let  $B$  be a finite type scheme over a field  $k$ , and assume that  $B$  is separated and normal. Let  $f : X \rightarrow B$  be a finite type morphism. A *PAC section* of  $f$  is an integral closed subscheme  $Y \subset E$  such that the restriction of  $f$ ,  $f_Y : Y \rightarrow B$ , is dominant with irreducible (but possibly nonreduced) geometric generic fiber. The *domain of definition* is the maximal open subscheme of  $B$  over which  $f_Y : Y \rightarrow B$  is faithfully flat.

**Corollary 2.4.** *For every finite type morphism  $f : X \rightarrow B$ , if  $f$  has no PAC section, then there exists a finite Galois extension  $k'/k$  and a dense open subset  $U \subset \text{Grass}_k(\mathbb{P}^r, \mathbb{P}_k^N)$  such that for every field extension  $K/k$  that is linearly disjoint from  $k'/k$ , for every  $[L] \in U(\text{Spec } K)$ , the restriction  $f_L : X \times_{\mathbb{P}_k^N} L \rightarrow B \times_{\mathbb{P}_k^N} L$  also has no PAC section.*

*Proof.* There exists a finite type, surjective monomorphism  $i : X' \rightarrow X$  such that every connected component  $X'_i$  of  $X'$  is regular and separated. Every PAC section of  $X' \rightarrow B$  maps under  $i$  to a PAC section of  $X \rightarrow B$ . Conversely, for every PAC section  $Y$  of  $X \rightarrow B$ , the generic point  $\eta_Y$  lifts uniquely to  $X'$ , and the closure of this generic point in  $X'$  gives a PAC section of  $X' \rightarrow B$ . The same argument holds for  $X' \times_{\mathbb{P}_k^N} L \rightarrow X \times_{\mathbb{P}_k^N} L \rightarrow B \times_{\mathbb{P}_k^N} L$ . Thus, up to replacing  $X$  by  $X'$ , assume that every connected component  $X_i$  of  $X$  is regular and separated, and assume that each restriction morphism  $f|_{X_i} : X_i \rightarrow B$  has no PAC section.

Up to shrinking  $B$ , assume that every  $f|_{X_i} : X_i \rightarrow B$  is flat, and that  $B$  is regular. The separable closure of  $k(B)$  in  $k(X_i)$  is a finite, separable extension of  $B$ . Denote by  $h_i : B_i \rightarrow B$  the integral closure of  $B$  in this finite, separable extension. Since  $B$  is finite type over a field,  $B$  is excellent. Thus,  $h_i$  is finite. By construction,  $h_i$

is generically étale. Up to shrinking  $B$  further, assume that  $h_i$  is everywhere finite and étale.

Since  $X_i$  is regular, it is normal. Thus,  $f|_{X_i}$  factors through  $h_i$ , i.e., there exists a finite type, dominant morphism  $g_i : X_i \rightarrow B_i$  such that  $f|_{X_i}$  equals  $h_i \circ g_i$ . By construction, the geometric generic fiber of  $g_i$  is irreducible. By [Jou83, Théorème 4.10, 6.10], there exists a dense open subscheme of  $B_i$  over which  $X_i$  is faithfully flat with irreducible geometric fibers. Since  $B_i$  is finite over  $B$ , up to shrinking  $B$  further, assume that this dense open subscheme equals all of  $B_i$ . Thus, every PAC section of  $f|_{X_i}$  maps under  $g_i$  to a PAC section of  $h_i$ . Conversely, for every PAC section  $Z$  of  $h_i$ , the inverse image  $g_i^{-1}(Z)$  is a PAC section of  $f|_{X_i}$ . In particular, since  $f|_{X_i}$  has no PAC section, also  $h_i$  has no PAC section. Since  $h_i$  is finite, PAC sections are the same as inseparable sections. So  $h_i$  has no inseparable sections. Denote by  $h : B' \rightarrow B$  the disjoint union of the finitely many morphisms  $h_i$ .

By Theorem 2.2, there exists a finite Galois extension  $k'/k$  and a dense open subset  $U \subset \text{Grass}_k(\mathbb{P}^r, \mathbb{P}^N)$  such that for every field extension  $K/k$  that is linearly disjoint from  $k'/k$ , for every  $[L] \in U(\text{Spec } K)$ , also  $h_L$  has no inseparable section. Every PAC section of  $f_L : X \times_{\mathbb{P}_k^N} L \rightarrow B \times_{\mathbb{P}_k^N} L$  maps under  $g$  to a PAC section of  $h_L$ . Since  $h_L$  is finite and has no inseparable section, it has no PAC section. Thus,  $f_L$  has no PAC section.  $\square$

### 3. FIELDS THAT ADMIT RATIONAL POINTS ON SPECIALIZATIONS OF RATIONALLY CONNECTED VARIETIES

After James Ax introduced PAC fields, he asked whether every specialization of Fano hypersurfaces in  $\mathbb{P}^{n-1}$  over a perfect PAC field  $L$  has an  $L$ -point [Ax68, Problem 3]. A projective variety over a field  $F$  is *rationally connected*, resp. *separably rationally connected*, if for every algebraically closed field extension  $E/F$ , resp. for every separably closed field extension  $E/F$ , every pair of  $E$ -points of the variety is contained in the image of an  $E$ -morphism from  $\mathbb{P}_E^1$ . In characteristic 0, these two definitions agree. Sufficiently general Fano hypersurfaces are *separably rationally connected varieties*, [KMM92] (in characteristic 0), [Zhu11] (in arbitrary characteristic). Thus, Ax was asking about rational points on specializations of certain separably rationally connected varieties.

The most common separably rationally connected varieties have unobstructed deformations, e.g., all Fano manifolds in characteristic 0, (standard) projective homogeneous spaces in all characteristics, Fano complete intersections of ample divisors in (standard) projective homogeneous spaces in all characteristics, etc. However, to formulate results that also apply to those rationally connected varieties with obstructed deformations, it is necessary to address *ramification*, particularly in mixed characteristic. For DVRs  $(\Lambda, \mathfrak{m}_\Lambda)$  and  $(R, \mathfrak{m}_R)$ , a local homomorphism  $\phi : \Lambda \rightarrow R$  is *regular* if

- (i)  $\phi(\mathfrak{m}_\Lambda)R$  equals  $\mathfrak{m}_R$ , i.e.,  $\phi$  is *weakly unramified* (note that this implies that  $\phi$  is injective),
- (ii) the residue field extension  $\Lambda/\mathfrak{m}_\Lambda \rightarrow R/\mathfrak{m}_R$  is separable (note that this holds automatically if  $\Lambda/\mathfrak{m}_\Lambda$  is perfect), and
- (iii) the fraction field extension is separable (note that this holds automatically if the fraction field has characteristic 0).



If the local homomorphism is essentially of finite type and if  $\Lambda$  is complete, then the first two hypotheses imply the third. The importance of regularity here has to do with *smooth parameter spaces*.

**Definition 3.1.** For an integral scheme  $S$ , a *parameter space* over  $S$  is a triple  $(M \rightarrow S, f_M : X_M \rightarrow M, \mathcal{L})$  of a smooth  $S$ -scheme  $M$  of pure relative dimension  $m$ , a flat, projective morphism  $f_M$ , and an invertible sheaf  $\mathcal{L}$  on the fiber of  $X_M$  over  $\text{Spec } \text{Frac}(S)$ .

**Lemma 3.2.** Let  $(R, \mathfrak{m}_R)$  be a DVR, let  $X_R$  be a flat, projective  $R$ -scheme, and let  $\mathcal{L}_{\text{Frac}(R)}$  be an invertible sheaf on  $X_{\text{Frac}(R)}$ . Let  $(\Lambda, \mathfrak{m}_\Lambda) \rightarrow (R, \mathfrak{m}_R)$  be a local homomorphism of DVRs that is regular. There exists a parameter space over  $S = \text{Spec } \Lambda$ , and there exists a dominant  $S$ -morphism  $\zeta : \text{Spec } R \rightarrow M$  such that  $(X_R, \mathcal{L}_R)$  is the pullback by  $\zeta$  of  $(X_M, \mathcal{L})$ .

*Proof.* For any subring  $L$  of  $R$ , by the usual limit arguments, there exists a subring  $A \subset R$  containing  $L$  such that  $L \rightarrow A$  is finitely generated, there exists  $X_A \rightarrow A$  a flat, projective scheme, and there exists an invertible sheaf on  $X_{\text{Frac}(A)}$ . Without loss of generality, also assume that  $A$  contains a generator for the principal ideal  $\mathfrak{m}_R \subset R$ . Now, set  $L$  equal to  $\phi(\Lambda)$ . Since  $\text{Frac}(A)$  is a subextension of  $\text{Frac}(\Lambda) \rightarrow \text{Frac}(R)$ , which is a separable extension by (iii), also  $\text{Frac}(\Lambda) \rightarrow \text{Frac}(A)$  is separably generated. Since  $A \otimes_\Lambda \text{Frac}(\Lambda)$  is finitely generated over  $\text{Frac}(\Lambda)$  with separably generated fraction field, there exists  $a \in A \setminus \{0\}$  such that  $A[1/a] \otimes_\Lambda \text{Frac}(\Lambda)$  is a smooth algebra over  $\text{Frac}(\Lambda)$ . Every uniformizing element  $\pi$  of  $\Lambda$  is also a uniformizing element of  $R$  by (i). Thus, for  $a \in A \subset R$ , there exists an integer  $e \geq 0$  and there exists  $u \in R \setminus \mathfrak{m}_R$  such that  $a$  equals  $u\pi^e$ . Adjoining  $u$  to  $A$  does not change  $A \otimes_\Lambda \text{Frac}(\Lambda)$ . Thus, assume that  $u$  is in  $A$ . Then  $A[1/u] \otimes_\Lambda \text{Frac}(\Lambda)$  equals  $A[1/a] \otimes_\Lambda \text{Frac}(\Lambda)$ . Since  $u$  is in  $R \setminus \mathfrak{m}_R$ , also  $1/u$  is in  $R \setminus \mathfrak{m}_R$ . Thus, adjoin  $1/u$  to  $A$ , and assume that  $A \otimes_\Lambda \text{Frac}(\Lambda)$  is smooth over  $\text{Frac}(\Lambda)$ .

Since  $A$  is a finite type, flat  $\Lambda$ -algebra such that  $A \otimes_\Lambda \text{Frac}(\Lambda)$  is smooth over  $\text{Frac}(\Lambda)$ , and by hypotheses (i) and (ii), there exists a Néron desingularization, [Sta17, Tag 0BJ6]. Precisely, there exist finitely many “Néron blowups”,  $A \mapsto A[\mathfrak{p}/\pi]$  where  $\mathfrak{p}$  equals  $\mathfrak{m}_R \cap A$ , after which  $\Lambda \rightarrow A$  is smooth at  $\mathfrak{p}$ , i.e.,  $A/\mathfrak{m}_A A$  is smooth over  $\Lambda/\mathfrak{m}_A$  at the prime  $\mathfrak{p}/\mathfrak{m}_A A$ . Thus, there exists  $v \in A \setminus \mathfrak{p}$  such that  $A[1/v]$  is smooth over  $\Lambda$ . Since  $v$  is in  $R \setminus \mathfrak{m}_R$ ,  $1/v$  is also in  $R \setminus \mathfrak{m}_R$ . Thus,  $A[1/v]$  is a subring of  $R$ . So after replacing  $A$  by  $A[1/v]$ , now  $A$  is a subring of  $R$  that is a finitely generated  $\Lambda$ -algebra that is smooth. Define  $M$  to be  $\text{Spec } A$ .  $\square$

**Definition 3.3.** A *prime finite DVR*  $(\Lambda, \mathfrak{m}_\Lambda)$  is a DVR whose residue field  $\Lambda/\mathfrak{m}_\Lambda$  is a finite extension of the prime subfield, i.e., either the residue field is a finite field if the characteristic is positive, or it is a number field if the characteristic is 0. A DVR  $(R, \mathfrak{m}_R)$  is *prime regular*, or *regular over a DVR whose residue field is finite over the prime subfield*, if there exists a prime finite DVR  $(\Lambda, \mathfrak{m}_\Lambda)$  and a local homomorphism  $\phi : \Lambda \rightarrow R$  that is regular.

**Lemma 3.4.** Every equicharacteristic DVR is prime regular.

*Proof.* Let  $(R, \mathfrak{m}_R)$  be an equicharacteristic DVR. Denote by  $F \subset R$  the prime subfield. Let  $\theta \in \mathfrak{m}_R$  be a generator. Since  $\theta$  is not an invertible element of  $R$ ,  $\theta$  is not in  $F$ . In fact,  $\theta$  is transcendental over  $F$ , for otherwise a minimal polynomial  $m_\theta(t) = t^d + \cdots + a_1\theta + a_0$  has degree  $d \geq 1$  and gives a relation

$1 = -a_0^{-1}\theta(a_1 + \cdots + \theta^{d-1})$ . This implies that  $\theta$  is invertible in  $R$ . Thus,  $F[\theta]$  is a copy of the polynomial ring in  $R$ . Since the multiplicative system  $F[\theta] \setminus \theta F[\theta]$  is contained in  $R \setminus \mathfrak{m}_R$ ,  $R$  contains the ring of fractions  $\Lambda = F[\theta]_{(\theta)}$ . The inclusion of DVRs  $(\Lambda, \mathfrak{m}_\Lambda) \rightarrow (R, \mathfrak{m}_R)$  is a local homomorphism. It is weakly unramified since  $\theta \in \mathfrak{m}_\Lambda$  is a generator of  $\mathfrak{m}_R$ . Since the residue field  $F$  of  $\Lambda$  is perfect, every field extension of the residue field is separable. Thus, it only remains to check that  $K = \text{Frac}(R)$  is separable over  $F(\theta)$ .

In characteristic 0 this is automatic. Assume the characteristic equals  $p$ . Since  $F(\theta)^{1/p}$  equals  $F(\theta)[t]/\langle t^p - \theta \rangle$ , we need to prove that  $A = K[t]/\langle t^p - \theta \rangle$  contains no nonzero nilpotent  $\alpha$  with  $\alpha^p$  equal to 0. The  $K$ -vector space  $A$  is free with basis  $(1, t, \dots, t^{p-1})$ . So every element  $\alpha$  of  $A$  has a unique decomposition,

$$\alpha = a_0 + a_1 t + \cdots + a_{p-1} t^{p-1}.$$

Let  $\alpha$  be a nonzero element, i.e., some  $a_i$  is nonzero. Denote by  $e \in \mathbb{Z}$  the minimum of the valuations of those  $a_i \in K$  that are nonzero. Then up to replacing  $\alpha$  by  $\theta^{-e}\alpha$ , assume that every  $a_i$  is in  $R$ , and at least one  $a_i$  has valuation 0. Let  $\ell$  be the minimal  $i$  with  $0 \leq i \leq p-1$  such that  $a_i$  has valuation 0. Then  $a_\ell$  is invertible, so that also  $a_\ell^p$  is invertible. Therefore  $a_\ell^p \theta^\ell$  has valuation  $\ell$ .

On the other hand, for every  $m$  with  $0 \leq m \leq p-1$  and  $m \neq \ell$ , either  $a_m$  equals 0 so that  $a_m^p \theta^m$  equals 0, or  $\text{val}(a_m) > 0$  so that  $a_m^p \theta^m$  has valuation  $\geq p > \ell$ , or  $\text{val}(a_m)$  equals 0 but  $m > \ell$  so that again  $a_m^p \theta^m$  has valuation  $m > \ell$ . So also the sum

$$\sum_{0 \leq m \leq p-1, m \neq \ell} a_m^p \theta^m,$$

is either zero or has valuation  $\geq \ell + 1$ . Thus the full sum,

$$\alpha^p = a_\ell^p \theta^\ell + \sum_{0 \leq m \leq p-1, m \neq \ell} a_m^p \theta^m$$

is nonzero of valuation  $\ell$ . Therefore,  $A$  contains no nonzero nilpotent elements, and  $K$  is separable over  $F(\theta)$ .  $\square$

Because of the lemma, the only DVRs that are not prime regular are mixed characteristic DVRs that are not weakly unramified over a Cohen ring, e.g., for  $e > 1$ , the localization of  $\mathbb{Z}[x, y]/\langle y^e - px \rangle$  at the height one prime generated by  $p$  and  $y$ .

For a DVR  $(\Lambda, \mathfrak{m}_\Lambda)$ , for a regular local homomorphism  $(\Lambda, \mathfrak{m}_\Lambda) \rightarrow (R, \mathfrak{m}_R)$ , for every pair  $(X_R \rightarrow \text{Spec } R, \mathcal{L}_{\text{Frac}(R)})$  of a flat, projective  $R$ -scheme  $X_R$  and an invertible sheaf on the generic fiber, there exists a parameter space  $(M \rightarrow \text{Spec } \Lambda, f_M : X_M \rightarrow M, \mathcal{L})$  and a  $\Lambda$ -morphism  $\zeta : \text{Spec } R \rightarrow M$  pulling back  $(X_M, \mathcal{L})$  to  $(X_R, \mathcal{L}_{\text{Frac}(R)})$ , by Lemma 3.2. Thus every extension field of  $R/\mathfrak{m}_R$  admits a morphism to  $M_0 = M \times_{\text{Spec } \Lambda} \text{Spec } (\Lambda/\mathfrak{m}_\Lambda)$ , and this morphism is even dominant.

For a DVR  $(\Lambda, \mathfrak{m}_\Lambda)$  and for a parameter space as above, let  $F$  be a field (not necessarily finite), let  $E$  be the function field of a geometrically integral  $F$ -scheme of dimension  $d$ , and let  $z : \text{Spec } E \rightarrow M_0$  be a morphism, not necessarily dominant. Let  $M_\eta^\circ$  be a specified dense open subset of the generic fiber  $M_\eta = M \times_{\text{Spec } \Lambda} \text{Spec } \text{Frac}(\Lambda)$  (for instance the entire generic fiber, but there are applications when  $M_\eta^\circ$  is a smaller dense open).

**Definition 3.5.** An *integral extension* of  $z$  is a triple

$$(B \rightarrow \operatorname{Spec} \Lambda, \pi_B^o : C_B^o \rightarrow B, z_B : C_B^o \rightarrow M),$$

and a pair

$$(\psi_B : \operatorname{Frac}(B_0) \rightarrow F, \psi_E : F(C_F^o) \rightarrow E)$$

consisting of a smooth, quasi-projective, surjective morphism  $B \rightarrow \operatorname{Spec} \Lambda$  with integral closed fiber and generic fiber, a smooth, quasi-projective, surjective morphism  $\pi_B^o$  of relative dimension  $d$  with geometrically integral fibers, and a  $\Lambda$ -morphism  $z_B$  together with a field homomorphism  $\psi_B : \operatorname{Frac}(B_0) \rightarrow F$  for the fraction field of the integral scheme  $B_0 = B \otimes_\Lambda \Lambda/\mathfrak{m}_\Lambda$  and an isomorphism of  $F$ -extensions  $\psi_C : F(C_F^o) \rightarrow E$  for the fraction field of the integral scheme  $C_F^o = C_B^o \times_B \operatorname{Spec} F$  such that

- (i)  $z_B^{-1}(M^o)$  contains the generic fiber  $C_B^o \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} \operatorname{Frac}(\Lambda)$
- (ii) the morphism  $z$  equals the composition of  $\operatorname{Spec} \psi : \operatorname{Spec} E \rightarrow \operatorname{Spec} F(C_F)$  and the morphism  $\operatorname{Spec} F(C_F^o) \rightarrow M$  induced by  $z_B$ .

**Remark 3.6.** For an integral extension, the stalk  $R$  of the structure sheaf of  $B$  at the generic point of the closed fiber  $B_0 = B \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} (\Lambda/\mathfrak{m}_\Lambda)$  is a DVR that is regular over  $\Lambda$  (and also essentially of finite type) since  $B$  is smooth over  $R$ .

**Lemma 3.7.** For a prime finite DVR  $(\Lambda, \mathfrak{m}_\Lambda)$ , for a parameter space over  $\Lambda$  and a specified dense open subset  $M_\eta^o$  of the generic fiber, for every pair  $(E/F, z : \operatorname{Spec} E \rightarrow M)$  as above, there exists an integral extension.

*Proof.* Notice first, since  $F$  is algebraically closed in  $E$ , the algebraic closure in  $E$  of the prime subfield is actually a subfield of  $F$ . Thus, since  $\Lambda/\mathfrak{m}_\Lambda$  is a finite extension of the prime field, the subfield  $\Lambda/\mathfrak{m}_\Lambda$  of  $E$  is actually a subfield of  $F$ . By limit arguments there exists a subring  $A_0 \subset F$  and a smooth morphism  $\pi_{B_0}^o : C_{B_0}^o \rightarrow \operatorname{Spec} A_0$  such that  $E$  equals  $F(C_F^o)$  and such that  $\Lambda/\mathfrak{m}_\Lambda \rightarrow A_0$  is finite type. Up to adjoining finitely many elements of  $F$  to  $A_0$ , up to replacing  $C_{B_0}^o$  by the base change over this larger ring  $A_0$ , and up to replacing  $C_{B_0}^o$  by a dense Zariski open subscheme, there also exists a  $\Lambda$ -morphism  $z_{B_0} : C_{B_0}^o \rightarrow M$  that induces  $z$ . Also, since  $\Lambda/\mathfrak{m}_\Lambda$  is perfect, up to inverting one nonzero element of  $A_0$ , the scheme  $B_0 = \operatorname{Spec} A_0$  is smooth over  $\Lambda/\mathfrak{m}_\Lambda$ .

It remains to extend from  $\Lambda/\mathfrak{m}_\Lambda$  to all of  $\Lambda$  the triple  $(B_0 = \operatorname{Spec} A_0, \pi_{B_0}^o : C_{B_0}^o \rightarrow B_0, z_{B_0} : C_{B_0}^o \rightarrow M)$ . This follows by the method of [SX, Section 3]. First, since  $A_0$  is a smooth, finite type algebra over  $\Lambda/\mathfrak{m}_\Lambda$ , up to a further localization, it is the quotient of a polynomial ring over  $\Lambda/\mathfrak{m}_\Lambda$  by an ideal generated by a regular sequence. Lifting the coefficients of the polynomials in this regular sequence, there exists a  $\Lambda$ -smooth algebra  $A'$  with  $A' \otimes_\Lambda \Lambda/\mathfrak{m}_\Lambda$  equal to  $A_0$ . If  $d$  equals 0, define  $C' \rightarrow \operatorname{Spec} A'$  to be the identity. For  $d \geq 1$ , up to localizing  $A_0$  and replacing  $C_{B_0}^o$  by a dense Zariski open, realize  $C_{B_0}^o$  as a dense open subset of a hypersurface in  $\mathbb{P}_{A_0}^{d+1}$ . For a general lift to  $A'$  of the coefficients of the defining polynomial of this hypersurface, the lift of the hypersurface in  $\mathbb{P}_{A'}^{d+1}$  is flat over a dense open subset of  $\operatorname{Spec} A'$  that contains  $\operatorname{Spec} A_0$ , and it is smooth over a dense open subset. Up to localizing  $A_0$  and  $A'$  further, this hypersurface is  $A'$ -flat. Since the geometric generic point is a smooth hypersurface of dimension  $d \geq 1$  in projective space, it is integral. Define  $C'$  to be a dense open subset that intersects the fiber over  $\operatorname{Spec} A_0$

and that is smooth over  $\text{Spec } A'$ . Since the geometric generic fiber is integral, up to shrinking  $C'$  further,  $C'$  has geometrically integral fibers over  $\text{Spec } A'$ .

Now consider the graph of  $z_{B_0}$  as a closed subscheme of the fiber product  $C' \times_{\text{Spec } \Lambda} M$ . Up to shrinking further, it is an irreducible component (of multiplicity 1) of a complete intersection of ample divisors in the closed fiber of  $C' \times_{\text{Spec } \Lambda} M$ . As above, lift the coefficients of the defining equations to lift  $z_{B_0}$  to a closed subscheme curve  $C$  of  $C' \times_{\text{Spec } \Lambda} M$  whose generic fiber over  $\text{Frac}(\Lambda)$  is a general complete intersection of ample divisors. Since  $\text{Frac}(\Lambda)$  is an infinite field, for an appropriate choice of the lifts of the coefficients,  $C$  is smooth and the image in  $M_\eta$  intersects  $M_\eta^o$ .

There is an issue about irreducibility of geometric fibers. Choosing projective models over  $\Lambda$  of all of the schemes, the Stein factorization  $\overline{B}$  of  $C \rightarrow \text{Spec } A'$  (roughly the integral closure of  $A'$  in the fraction field of  $C$ ) may be nontrivial. However, since the closed fiber  $C_0$  is smooth over  $A_0$  with geometrically integral fibers,  $\text{Spec } A_0$  is an irreducible component of the closed fiber  $\overline{B}_0$ . Replace  $\overline{B}$  by the open complement in  $\overline{B}$  of the union of the finitely many irreducible components of  $\overline{B}_0$  different from  $\text{Spec } A_0$ . The restriction of  $C$  over  $\overline{B}$  now has geometrically irreducible fibers. Define  $C_B^o$  to be the open subset of  $C$  that is the smooth locus of the morphism to  $\overline{B}$ . Define  $B$  to be the open image in  $\overline{B}$  of this smooth morphism. Define  $z_B$  to be the restriction to the locally closed subscheme  $C_B^o$  of  $C \times_{\text{Spec } \Lambda} M$  of the projection to  $M$ . By construction, the inverse image of  $M_\eta^o$  in the generic fiber  $C_\eta^o = C_B^o \otimes_{\text{Spec } \Lambda} \text{Spec } \text{Frac}(\Lambda)$  is a dense open. The complement is a proper closed subset  $D_\eta$ . Since  $C_B^o$  is flat over  $\text{Spec } \Lambda$ , the closure  $D$  of  $D_\eta$  is flat over  $\text{Spec } \Lambda$ . Thus,  $D_\eta$  cannot contain the irreducible component  $C_{B_0}^o$ . After replacing  $C_B^o$  by the open complement of  $D_\eta$ , this triple  $(B \rightarrow \text{Spec } \Lambda, \pi_B^o : C_B^o \rightarrow B, z_B : C_B^o \rightarrow M)$  is an integral extension of  $z$ .  $\square$

Here is a precise formulation of existence of rational points for specializations of separably rationally connected varieties.

**Definition 3.8.** A field  $L$  is *RC solving*, or *admits rational points on specializations of separably rationally connected varieties*, if for every projective, flat scheme  $X_R$  over a prime regular DVR  $R$  such that  $X_R \times_{\text{Spec } R} \text{Spec } \overline{\text{Frac}(R)}$  is smooth, integral, and separably rationally connected, for every field extension  $z^* : R/\mathfrak{m}_R \hookrightarrow L$ ,  $X_R \times_{\text{Spec } R} \text{Spec } L$  has an  $L$ -rational point. The field  $L$  is *characteristic 0 RC solving* if the condition holds for every prime regular DVR  $R$  whose fraction field has characteristic 0.

**Remark 3.9.** To summarize the lemmas above, the prime regular hypothesis is automatic if  $(R, \mathfrak{m}_R)$  is an equicharacteristic DVR. Also,  $L$  is RC solving if and only if, for every prime finite DVR  $(\Lambda, \mathfrak{m}_\Lambda)$ , for every parameter space over  $\text{Spec } \Lambda$  such that there is a dense open subset  $M_\eta^o$  of the generic fiber over which  $f_M$  is smooth with geometric fibers that are integral and separably rationally connected, for every morphism to the closed fiber  $\text{Spec } L \rightarrow M_0$ , the pullback  $X_M \times_M \text{Spec } L$  has an  $L$ -point.

**Theorem 3.10.** [GHMS05, Lemma 2.5] *For every algebraically closed field  $k$ , every function field  $k(C)$  of a geometrically integral, smooth, projective  $k$ -curve  $C$  is RC solving.*

*Proof.* The notation is as in the definition. By hypothesis there exists a prime finite DVR  $(\Lambda, \mathfrak{m}_\Lambda)$  and a local homomorphism  $(\Lambda, \mathfrak{m}_\Lambda) \rightarrow (R, \mathfrak{m}_R)$ . By Lemma 3.2, there exists a parameter space  $(M \rightarrow S, f_M : X_M \rightarrow M, \mathcal{L})$  and a  $\Lambda$ -morphism  $\zeta : \text{Spec } R \rightarrow M$  pulling back  $X_M$  to  $X_R$ . In particular, the geometric generic fiber  $X_R \otimes_R \overline{\text{Frac}(R)}$  is the base change of the geometric generic fiber of  $f_M$ . Each of the following properties of a proper scheme over an algebraically closed field holds if and only if it holds after base change to an arbitrary algebraically closed field extension: smoothness, integrality, separable rational connectedness. Thus, these properties all hold for the geometric generic fiber. Thus the generic point of  $M$  is contained in the maximal open subscheme  $M_\eta^o$  of the generic fiber  $M_\eta = M \otimes_\Lambda \text{Frac}(\Lambda)$  over which the geometric fibers of  $f_M$  are smooth, integral, and separably rationally connected. So  $M_\eta^o$  is a dense open subset of the generic fiber  $M_\eta$ .

Define  $z : \text{Spec } k(C) \rightarrow M$  to be the composition of  $\text{Spec } k(C) \rightarrow \text{Spec } R/\mathfrak{m}_R$  and the restriction to  $\text{Spec } R/\mathfrak{m}_R$  of  $\zeta$ . By Lemma 3.7, there exists an integral extension of  $z$ . Denote by  $f_C : X_C \rightarrow C_B^o$  the pullback of  $X_M$  by  $z_B$ . By construction,  $\pi_B^o : C_B^o \rightarrow B$  is smooth, quasi-projective of relative dimension 1, and the geometric generic fiber of  $f_C$  is a smooth, integral, separably rationally connected variety. Thus, by [GHS03], [dJS03], there exists a finite extension  $K'$  of the fraction field of  $B$  and a  $C_B^o$ -morphism  $s : \text{Spec } K' \times_B C_B^o \rightarrow X_C$ . Consider the DVR  $(\mathcal{O}, \mathfrak{n})$  that is the stalk  $\mathcal{O}_{B, \eta_{B_0}}$  of the structure sheaf of  $B$  at the generic point of the closed fiber  $B_0 = B \times_{\text{Spec } \Lambda} \text{Spec } (\Lambda/\mathfrak{m}_\Lambda)$ . By the Krull-Akizuki theorem, there exists a DVR  $(\mathcal{O}', \mathfrak{n}')$  with fraction field  $K'$  that dominates  $(\mathcal{O}, \mathfrak{n})$  (but  $\mathcal{O}'$  is not necessarily a finite  $\mathcal{O}$ -module). Since  $f_C$  is proper, by the valuative criterion of properness, the maximal domain  $V$  of definition of the rational transformation  $s : \text{Spec } \mathcal{O}' \times_B C_B^o \rightarrow X_C$  intersects the closed fiber  $\text{Spec } (\mathcal{O}'/\mathfrak{n}') \times_B C_B^o \rightarrow X_C$ . Thus,  $s$  gives a rational section of  $f_C$  over the base change of  $C_{B_0}^o$  to the extension field  $\mathcal{O}'/\mathfrak{n}'$  of the fraction field  $\text{Frac}(B_0)$ . So the same holds after extension to any bigger field, e.g., algebraic closure of  $\mathcal{O}'/\mathfrak{n}'$ .

Existence of a section of a morphism of finite type schemes over an algebraically closed field is a property that holds if and only if it holds after an arbitrary extension to an algebraically closed field. Thus, since it holds after extension from  $\text{Frac}(B_0)$  to the algebraic closure of  $\mathcal{O}'/\mathfrak{n}'$ , it also holds after extension from  $\text{Frac}(B_0)$  to  $k$ . Therefore there is a  $k(C)$ -point of  $X_R \otimes_R k(C)$ .  $\square$

**Theorem 3.11.** [Esn07], [EX09] *Every finite field is RC solving.*

**Remark 3.12.** Every subfield of a finite field is a finite field. So for a finite field  $L$ , for every DVR  $(R, \mathfrak{m}_R)$  and field homomorphism  $R/\mathfrak{m}_R \rightarrow L$ , already  $(R, \mathfrak{m}_R)$  is prime finite. So  $(R, \mathfrak{m}_R)$  is prime regular.

*Proof.* When  $R$  has mixed characteristic, this follows from [Esn07]. When  $R$  is equicharacteristic, this follows from [EX09].  $\square$

**Theorem 3.13.** [HX09] *Every PAC field of characteristic 0 is RC solving.*

**Theorem 3.14.** [Sta13, Theorem 1.1] *A PAC field of characteristic  $p$  is RC solving if it contains a primitive root of unity of order  $n$  for every integer  $n$  prime to the characteristic.*

**Remark 3.15.** Please note: in [Sta13] the hypothesis that the DVR is prime regular is missing. This is a mistake. The proof is only valid under the hypothesis

that the DVR is prime regular: for the “bifurcation” in Step 3 in Section 2 and the use of [GHMS05, Lemma 2.5] in the argument preceding Lemma 1.12, the argument holds only if the local homomorphism  $S_{\mathfrak{m}_S} \rightarrow \mathcal{O}_{P,Q}$  is weakly unramified. For Corollary 1.2, and similar applications, this is irrelevant: there is a parameter space over  $\text{Spec } \mathbb{Z}$  for Fano complete intersections.

*Proof.* The proof in [Sta13] depends on a symmetry of “Bertini theorems”. Here is a proof that instead uses the finite field Bertini theorem.

By Lemma 3.2 and Lemma 3.7 (with  $d = 0$ ) assume we have the following: a DVR  $(\Lambda, \mathfrak{m}_\Lambda)$  whose residue field is a finite field, a quasi-projective, smooth  $\Lambda$ -scheme  $B$  of relative dimension  $m$ , a projective, flat morphism  $g_B : X_B \rightarrow B$ , and a field homomorphism  $\text{Frac}(B_0) \hookrightarrow L$  from the fraction field of the (integral) closed fiber  $B_0 = B \times_{\text{Spec } \Lambda} \text{Spec } (\Lambda/\mathfrak{m}_\Lambda)$  to a perfect PAC field  $L$  that is nice. Replace  $\Lambda$  by its Henselization, and base change  $B$  and  $X_B$  over the Henselization. Then there is a unique connected component of  $B$  that contains  $B_0$ . Up to replacing  $B$  by this open and closed subset, assume that  $B \rightarrow \text{Spec } \Lambda$  has irreducible geometric generic fiber.

Since  $B$  is smooth over  $\text{Spec } \Lambda$ , up to replacing  $B$  by an open subset that is dense in all fibers, there exists an étale morphism  $f : B \rightarrow \mathbb{A}_\Lambda^m$ . Denote  $X_B \times_B B_0$  by  $X_{B_0}$ . By the proof of Corollary 2.4, there exists a finite type, surjective monomorphism  $i : X'_0 \rightarrow X_0$  such that every connected component of  $X'_0$  is regular and separated. Up to shrinking  $B$  and  $B_0$ , assume that the composition  $g_0 \circ i : X'_0 \rightarrow B_0$  is flat. For each of the finitely many connected components, the algebraic closure of the field  $\Lambda/\mathfrak{m}_\Lambda$  in the function field of the component is a finite extension obtained by adjoining a root of unity whose order  $n$  is prime to  $p$ . Thus the compositum of these fields is obtained by adjoining a root of unity  $\alpha$ .

Denote by  $m_\alpha(t)$  the minimal polynomial of  $\alpha$  over  $\Lambda/\mathfrak{m}_\Lambda$ . This is an irreducible, separable, monic polynomial. Let  $m(t) \in \Lambda[t]$  be any element reducing to  $m_\alpha(t)$ . Then  $\Lambda' = \Lambda[t]/\langle m(t) \rangle$  is an étale extension of  $\Lambda$ . Since the quotient  $\Lambda'/\mathfrak{m}_\Lambda \Lambda'$  is a field, the ideal  $\mathfrak{m}_{\Lambda'} = \mathfrak{m}_\Lambda \Lambda'$  is a maximal ideal. Since  $\mathfrak{m}_\Lambda$  is principal, also  $\mathfrak{m}_{\Lambda'}$  is principal. Finally, by hypothesis the field  $L$  contains all roots of unity, so  $\Lambda/\mathfrak{m}_\Lambda \rightarrow L$  factors through  $\Lambda'/\mathfrak{m}_{\Lambda'}$ . Thus, up to replacing  $\Lambda$  by  $\Lambda'$  and replacing every scheme as above by its base change by the finite and regular local homomorphism  $\Lambda \rightarrow \Lambda'$ , assume that  $\Lambda/\mathfrak{m}_\Lambda$  is algebraically closed in the fraction field of every irreducible component of  $X'_0$ . Thus the field extension in Corollary cor-ffBertini is an isomorphism. So every field extension of  $\Lambda/\mathfrak{m}_\Lambda$  satisfies the hypothesis of the corollary.

Denote by  $k$  the residue field  $\Lambda/\mathfrak{m}_\Lambda$ . If  $X'_0 \rightarrow B_0$  has a PAC section  $Y_0 \subset X'_0$ , then the base change  $Y_0 \times_{B_0} \text{Spec } L$  is a geometrically irreducible, finite type  $L$ -scheme. Since  $L$  is a perfect PAC field, this scheme has an  $L$ -point, which then maps under  $i$  to an  $L$ -point of  $X_0 \times_{B_0} \text{Spec } L$ , as desired.

Thus, by way of contradiction, assume that there is no PAC section of  $g$ . Then by Corollary 2.4 with  $k'$  equal to  $k$ , there exists a dense, Zariski open subset  $U \subset \text{Grass}_k(\mathbb{P}^1, \mathbb{P}^m)$  such that for every field extension  $K/k$  and for every  $[L] \in U(K)$ , the inverse image  $f^{-1}(L)$  is geometrically irreducible, and the restriction of  $g$  over  $f^{-1}(L)$  has no PAC section. By hypothesis, the family  $X_B \rightarrow B \rightarrow \text{Spec } \Lambda$  is a parameter space such that the geometric generic fiber of  $g_B$  is smooth, integral,

and separably rationally connected. Thus, by Theorem 3.10, up to replacing  $K$  by a finite extension  $K'/K$  obtained by adjoining a root of unity, there is a rational section of the restriction of  $g$  over  $f^{-1}(L)$ . This contradiction implies that  $g$  does have a PAC section. Therefore  $X_0 \times_{B_0} \text{Spec } L$  has an  $L$ -point.  $\square$

#### 4. SPECIALIZATIONS OF RATIONALLY SIMPLY CONNECTED VARIETIES

The first new result is an analogous result for perfect PAC fields of positive characteristic that do not necessarily contain all roots of unity, but where rational connectedness is replaced by *rational simple connectedness*. For an algebraically closed field  $\overline{Q}$  of characteristic 0, a *rational simple connected fibration* over a  $\overline{Q}$ -curve is a pair  $(f_{\overline{Q}} : X_{\overline{Q}} \rightarrow C_{\overline{Q}}, \mathcal{L}_{\overline{Q}})$  where  $C_{\overline{Q}}$  is a smooth, projective, connected  $\overline{Q}$ -curve, where  $f_{\overline{Q}}$  is a proper, flat morphism, and where  $\mathcal{L}_{\overline{Q}}$  is an invertible sheaf that satisfies the following six hypotheses. First,  $X_{\overline{Q}}$  is smooth over  $\overline{Q}$ . Second, every geometric fiber of  $f_{\overline{Q}}$  is irreducible. Third,  $\mathcal{L}_{\overline{Q}}$  is  $f_{\overline{Q}}$ -ample. The final three hypotheses involve the geometric generic fiber  $Y$  of  $f_{\overline{Q}}$  is a scheme over the algebraically closure  $k$  of the fraction field of  $C_{\overline{Q}}$ , together with the pullback  $\mathcal{L}_Y$  of  $\mathcal{L}_{\overline{Q}}$  as an invertible sheaf on  $Y$ . The fourth hypothesis is that for the parameter space  $\overline{\mathcal{M}}_{0,1}(Y/k, 1)$  of 1-pointed, genus 0 stable maps to  $Y$  having  $\mathcal{L}_Y$ -degree 1, i.e., “lines”  $\ell$ , for the maximal open subscheme  $Y_{\text{free}}$  of  $Y$  over which the evaluation morphism

$$\text{ev}_1 : \overline{\mathcal{M}}_{0,1}(Y/k, 1) \rightarrow Y, \quad ([\ell], p) \mapsto p,$$

is smooth (automatically this is a dense open in  $Y$ ), the fiber of  $\text{ev}_1$  over every geometric point of  $Y_{\text{free}}$  is nonempty, irreducible, and has rationally connected fibers. There is a second important morphism, the “forgetful” morphism,

$$\Phi : \overline{\mathcal{M}}_{0,1}(Y/k, 1) \rightarrow \overline{\mathcal{M}}_{0,0}(Y/k, 1), \quad ([\ell], p) \mapsto [\ell].$$

For every integer  $m \geq 1$  there is a parameter  $k$ -scheme  $\text{FreeChains}_2(Y/k, m)$  of ordered  $m$ -tuples

$$(([\ell_1], p_{1,0}, p_{1,\infty}), \dots, ([\ell_m], p_{m,0}, p_{m,\infty}))$$

of triples  $([\ell_i], p_{i,0}, p_{i,\infty})$  of 2-pointed lines in  $Y$  such that  $p_{i,0}$  and  $p_{i,\infty}$  are in the dense open  $Y_{\text{free}}$ , and such that  $p_{i,\infty}$  equals  $p_{i+1,0}$  as points of  $Y_{\text{free}}$  for every  $i = 1, \dots, m-1$ . There is an evaluation morphism

$$\text{ev}_2 : \text{FreeChains}_2(Y/k, m) \rightarrow Y \times_{\text{Spec } (k)} Y,$$

that sends each ordered  $m$ -tuple as above to the ordered pair  $(p_{1,0}, p_{m,\infty})$ . The fifth hypothesis is that there exists an integer  $m \geq 1$  and a dense open  $V$  of  $Y \times_{\text{Spec } (k)} Y$  such that the fiber of  $\text{ev}_2$  over every geometric point of  $V$  is nonempty, irreducible, and “birationally rationally connected”, i.e., there exists one projective model of this quasi-projective variety that is rationally connected (hence every projective model is rationally connected). If this holds for one  $m$ , then there exists an integer  $m_0$  such that it holds for every  $m \geq m_0$ . The final hypothesis is that  $(Y, \mathcal{L}_Y)$  contains a very twisting scroll, cf. [dJHS11, Definition 12.7]. This hypothesis is equivalent to existence of a morphism  $\zeta : \mathbb{P}_k^1 \rightarrow \overline{\mathcal{M}}_{0,1}(Y/k, 1)$  such that all of the following hold,

- (i) the composition  $\text{ev}_1 \circ \zeta : \mathbb{P}_k^1 \rightarrow Y$  is free, i.e.,  $(\text{ev}_1 \circ \zeta)^* T_{Y/k}$  is globally generated,
- (ii) the morphism  $\text{ev}_1$  is smooth at every point in  $\zeta(\mathbb{P}_k^1)$ ,

- (iii) the pullback by  $\zeta$  of the relative tangent sheaf  $T_{\text{ev}_1}$  is ample, and
- (iv) the pullback by  $\zeta$  of  $T_\Phi$  is globally generated.

For a characteristic 0 field  $Q$ , a pair  $(f_Q : X_Q \rightarrow C_Q, \mathcal{L}_Q)$  is a *rationally simply connected fibration* over a  $Q$ -curve if the base change of the pair to the algebraic closure  $\overline{Q}$  is a rationally simply connected fibration over a  $\overline{Q}$ -curve as above. The main theorem of [dJHS11], Theorem 13.1 (cf. also [Sta10, Definition 4.8 and Theorem 4.9]), gives an integer  $\epsilon$  and a sequence  $(Z_{Q,e})_{e \geq \epsilon}$  of irreducible components  $Z_{Q,e}$  of the Hilbert scheme  $\text{Hilb}_{X_Q/Q}^{et+1-g(C_Q)}$  satisfying all of the following.

- (i) The geometric generic point of  $Z_{Q,e}$  parameterizes the closed image of a section  $\sigma : C_Q \rightarrow X_Q$  of  $f_Q$  of  $\mathcal{L}_Q$ -degree  $e$  and that is  $(g)$ -free, i.e., the deformations of the section relative to a fixed divisor of degree  $\max(2g(C_Q), 1)$  are unobstructed.
- (ii) The restriction to  $Z_{Q,e}$  of the Abel map of  $\mathcal{L}_Q$ ,  $\alpha_{\mathcal{L}_Q}|_Z : Z_{Q,e} \rightarrow \text{Pic}_{C_Q/Q}^e$ , has fiber over the geometric generic point of  $\text{Pic}_{C_Q/Q}^e$  that is nonempty, irreducible, and rationally connected.
- (iii) After arbitrary base change from  $Q$  to an algebraically closed field, for every section  $\sigma$  as above that is  $(g)$ -free, after attaching to the closed image  $\sigma(C_Q)$  sufficiently many general lines in general fibers of  $f_Q$ , the resulting curve is parameterized by  $Z_{Q,e}$  for the appropriate  $\mathcal{L}_Q$ -degree  $e$ .

In order to state the result, it is useful to specify a bounded family of polarized schemes. Let  $S$  be an integral, Noetherian, regular scheme of dimension  $\leq 1$  whose function field  $Q$  has characteristic 0; usually  $S$  will be a dense Zariski open subset of the ring of integers  $\text{Spec } \mathfrak{o}_Q$  of a number field  $Q$ . Fix integers  $m, c \geq 1$ .

**Definition 4.1.** [SX, Definition 1.9] A *parameter datum* over  $S$  with a codimension  $> c$  compactification is a datum

$$((M, f_M : X_M \rightarrow M, \mathcal{L}_{M_Q}), (\overline{M}, \mathcal{O}_{\overline{M}}(1), i))$$

of a smooth  $S$ -scheme  $M$  of constant relative dimension  $m > 1$ , a proper, flat morphism  $f_M$ , an  $f_M$ -very ample invertible sheaf  $\mathcal{L}_{M_Q}$  on  $X \times_S \text{Spec}(Q)$ , and a codimension  $> c$  compactification  $(\overline{M}, \mathcal{O}_{\overline{M}}(1), i)$  of  $M$  over  $S$ , i.e.,  $q : \overline{M} \rightarrow S$  is flat and proper,  $\mathcal{O}_{\overline{M}}(1)$  is  $q$ -very ample,  $i : M \rightarrow \overline{M}$  is a dense open immersion such that  $\partial \overline{M} := \overline{M} \setminus M$  intersects every generic fiber of  $q$  in a subscheme of codimension  $> c$  in that fiber.

For every integer  $r \geq 1$ , the complete linear system  $H^0(\overline{M}_Q, \mathcal{O}_{\overline{M}_Q}(r))$  induces a closed immersion into projective space over  $\text{Spec}(Q)$ . For the corresponding Grassmannian parameterizing flat families of linear subvarieties of this projective space of codimension  $m - 1$ , there is a dense open subset  $G_r$  parameterizing linear subvarieties whose inverse images in  $\overline{M}_Q$  are smooth, irreducible curves that are complete contained in the open subset  $M_Q$ .

**Definition 4.2.** [SX, Definition 1.10] A parameter datum as above satisfies the *RSC property* if there exists an integer  $r_0$  and a sequence  $(W_r)_{r \geq r_0}$  of dense open subschemes  $W_r \subset G_r$  such that for every algebraically closed extension  $\overline{K}$  of  $Q$  and for every  $\overline{K}$ -point of  $W_r$  parameterizing a smooth curve  $C_{\overline{K}}$  in  $M_{\overline{K}}$ , the pullback family  $f : C_{\overline{K}} \times_M X \rightarrow C_{\overline{K}}$  together with the pullback of  $\mathcal{L}$  is a rationally simply connected fibration over  $C_{\overline{K}}$ .



The results of [SX] are stated with respect to a *finite field*  $F$ . However, the results hold for any RC solving field. When the field is the function field of a curve over an algebraically closed field, this is already in the proofs of [dJHS11, Corollary 13.2 and Lemma 13.4].

**Proposition 4.3.** [SX, Theorem 1.6] *Let  $(R, \mathfrak{m}_R)$  be a Henselian DVR that is prime regular with characteristic 0 fraction field  $K$ , and let  $R/\mathfrak{m}_R \rightarrow L$  be a field extension. Let  $C_R \rightarrow \operatorname{Spec} R$  be a generically smooth  $R$ -curve. Let  $f_R : X_R \rightarrow C_R$  be a projective, surjective morphism, and let  $\mathcal{L}_K$  be an  $f_R$ -ample invertible sheaf on the generic fiber  $X_K = X_R \times_{\operatorname{Spec} R} \operatorname{Spec} K$ . Let  $z : \operatorname{Spec} L \rightarrow \operatorname{Spec} R/\mathfrak{m}_R$  be a field extension that is RC solving. If  $(f_{\overline{K}} : X_{\overline{K}} \rightarrow C_{\overline{K}}, \mathcal{L}_{\overline{K}})$  is a rationally simply connected fibration, then there exists a sequence  $(U_e)_{e \geq \epsilon}$  of dense open subschemes  $U_e \subset \operatorname{Pic}_{C_K/K}^e$  such that  $X_L/C_L$  admits rational points compatibly with  $(U_e)_{e \geq \epsilon}$ . In particular, for every generic point  $\eta$  of the smooth locus of  $C_L = C_R \times_{\operatorname{Spec} R} \operatorname{Spec} L$ , the  $L(\eta)$ -scheme  $X_R \times_{C_R} \operatorname{Spec} L(\eta)$  has an  $L(\eta)$ -point.*

*Proof.* The proof in [SX] applies in the RC solving case. By [dJHS11, Theorem 13.1], there exists a sequence  $(Z_{R,e})_{e \geq \epsilon}$  of geometrically integral closed subschemes of the Hilbert scheme  $\operatorname{Hilb}_{X_R/R}$  whose geometric generic point parameterizes the image of a section of  $X_K \rightarrow C_K$  and such that the restriction of the natural Abel map  $\alpha_e : Z_{K,e} \rightarrow \operatorname{Pic}_{C_K/K}^e$  is surjective with geometric generic fiber an integral scheme that is rationally connected. Since  $K$  has characteristic 0, by resolution of singularities, there exists a projective morphism  $\tilde{Z}_{R,e} \rightarrow Z_{R,e}$  such that  $\tilde{Z}_{K,e}$  is  $K$ -smooth. Then  $U_e$  is defined as the maximal open subscheme over which  $\tilde{Z}_{K,e} \rightarrow \operatorname{Pic}_{C_K/K}^e$  is smooth and the fibers intersect the dense open parameterizing closed subschemes of  $X_K$  that are sections of  $X_K \rightarrow C_K$ . For every  $K$ -point  $[A]$  of  $U_e$  (and these do exist for  $e$  sufficiently positive and divisible using the fact that  $R$  is Henselian) the closure  $\tilde{Z}_{R,A}$  in  $\tilde{Z}_{R,e}$  of the fiber of  $\alpha_e$  over this  $K$ -point is a projective, flat scheme over  $\operatorname{Spec} R$  whose geometric generic fiber is smooth, integral and separably rationally connected. Thus, since  $L$  is RC solving, there exists an  $L$ -point of  $\tilde{Z}_{R,A} \times_{\operatorname{Spec} R} \operatorname{Spec} L$ . Using [SX, Lemma 4.1], this  $L$ -point gives an  $L(\eta)$ -point of  $X_R \times_{C_R} \operatorname{Spec} L(\eta)$ .  $\square$

The various complements to the main theorem from [SX] also extend.

**Proposition 4.4.** *Let  $S$  be a regular, integral, Noetherian scheme of dimension  $\leq 1$  whose function field has characteristic 0 and whose residue fields at closed points are either finite fields or characteristic 0 fields. Fix a parameter datum over  $S$  with a codimension  $> 1$  compactification. Then the following results of [SX] hold with the finite field  $F$  replaced by any RC solving field  $L$ : Proposition 1.12 and Corollary 1.13. Assuming that  $S$  is a dense open of the spectrum of the ring of integers of a number field, Corollary 1.14 also holds. Finally, for a parameter datum as above, for a closed point  $\mathfrak{s} \in S$  and a regular local homomorphism of DVRs  $\mathcal{O}_{S,\mathfrak{s}} \rightarrow R$ , for  $\pi_R : C_R \rightarrow \operatorname{Spec} R$ , for an  $S$ -morphism  $\zeta : \mathcal{O}_{C_R,\eta} \rightarrow \overline{M}$ , and for a projective flat morphism  $f_\eta : X_\eta \rightarrow \operatorname{Spec} \mathcal{O}_{C_R,\eta}$  as in Proposition 1.17, there exists a modification of the parameter datum over  $S$  such that  $\zeta$  extends to a regular morphism  $\zeta'$  to  $M'$  and such that the pullback by  $\zeta'$  of  $X'_{M'} \rightarrow M'$  equals  $f_\eta$ . In particular, if the parameter datum satisfies the RSC property, for every field homomorphism from the residue field of a closed point of  $S$  to  $L$ , for every function field  $E = L(\eta)$  of*

a geometrically integral  $L$ -curve, and for every  $S$ -morphism  $z_E : \operatorname{Spec} E \rightarrow M$ , the pullback  $X_E$  of  $X_M$  by  $z_E$  has an  $E$ -rational point.

*Proof.* The proof of Proposition 1.12 is essentially Corollary 3.15, which is valid over any field  $F$ .

The proof of Corollary 1.13 appears to use the fact that the field is finite, so that the image  $z_o$  under  $z : \operatorname{Spec} L[[t]] \rightarrow \overline{M}$  of the closed point is a closed point of  $\overline{M}$ . That can certainly fail if  $L$  is not a finite field. However, since the construction of the curve  $C_e$  is made with respect to the base change  $\overline{M}_L = \overline{M} \times_R \operatorname{Spec} L$ , the image point  $z_0$  in  $\overline{M}_L$  is an  $L$ -point, and that is a closed point since  $\overline{M}_L$  is a finite type  $L$ -scheme. Let  $X_{\overline{M}} \rightarrow \overline{M}$  be any projective model of  $X_M \rightarrow M \subset \overline{M}$ . By [Art69, Theorem 1.10], for every integer  $e$  there exists an integral closed curve  $C_e$  in  $\overline{M}_L \times_{\operatorname{Spec} L} \mathbb{P}_L^1$  that approximates to order  $e$  the graph of  $z : \operatorname{Spec} L[[t]] \rightarrow \overline{M}_L \times_{\operatorname{Spec} L} \mathbb{P}_L^1$ , considered as a formal section of  $\operatorname{pr}_2 : \overline{M}_L \times_{\operatorname{Spec} L} \mathbb{P}_L^1 \rightarrow \mathbb{P}_L^1$  over the closed point  $0 \in \mathbb{P}_L^1(\operatorname{Spec} L)$ . Since  $z$  maps the generic point of  $L[[t]]$  to  $M$ , also there exists such  $C_e$  that intersects the open  $M_L \times_{\operatorname{Spec} L} \mathbb{P}_L^1$  in a dense open  $C_e^o$ ; in fact this condition will be automatic for all  $e \gg 0$ . Proposition 1.12 then gives a section of the pullback  $X_M \times_M C_e^o \rightarrow C_e^o$ , which extends to a section of  $X_{\overline{M} \times \overline{M}} C_e \rightarrow C_e$  by the valuative criterion of properness. Then by [Gre66, Theorem 1], also  $X_{\overline{M} \times \overline{M}} \operatorname{Spec} L[[t]] \rightarrow \operatorname{Spec} L[[t]]$  has a section whose generic fiber is a section of  $X_M \times_M \operatorname{Spec} L((t)) \rightarrow \operatorname{Spec} L((t))$ .

Corollary 1.14 uses [Den16]. Although the article is focused on completions at closed points of  $\mathbb{Z}$ , resp.  $\mathbb{F}_p[[t]]$ , the proofs have no hypotheses that the rings involved should be finite over the residue field, etc. Using the beautiful general arguments, the key computation is the isomorphism  $\tau_p : \mathcal{MR}(\mathbb{Z}_p) \rightarrow \mathcal{MR}(\mathbb{F}_p[[t]])$ . This is completely general. Let  $(A, \theta A)$  and  $(B, \pi B)$  be Henselian DVRs, and let  $\overline{\tau} : A/\theta A \rightarrow B/\pi B$  be a field isomorphism. For every unit  $u \in A \setminus \theta A$ , there exists a unit  $v \in B \setminus \pi B$  such that  $\overline{\tau}(\overline{u})$  equals  $\overline{v}$ . Thus, there is a unique bijection  $\tau_{\theta, \pi} : \mathcal{MR}(A) \rightarrow \mathcal{MR}(B)$  such that for every integer  $n \geq 0$  and every unit  $u$ ,  $\tau_{\theta, \pi}(\operatorname{mres}(u\theta^n))$  equals  $\operatorname{mres}(v\pi^n)$ . This bijection is all that is used in the Transfer of Residues Lemma, and thus also in the Transfer of Surjectivity Theorem. In particular, this applies for  $A$  equal to  $L[[t]]$  with  $\theta$  to  $t$  and to  $B$  equal to a Cohen ring for  $L$  with  $\pi$  equal to  $p$  (the Witt vectors is one explicit construction of such a Cohen ring).

The proof of Proposition 1.17 works in the same way. In order to apply the Néron desingularization, it is important that the local homomorphism of DVRs is regular.  $\square$

To extend [SX, Proposition 1.18], we apply Lemma 3.7 to the parameter space of complete intersection curves in a specified parameter datum. Thus, let  $S$  be the spectrum of a Dedekind domain such that the fraction field  $Q$  of  $S$  has characteristic 0, and every residue field is a finite field. For a specified closed point of  $S$ , denote by  $(\Lambda, \mathfrak{m}_\Lambda)$  the stalk of the structure sheaf at that point. Thus  $(\Lambda, \mathfrak{m}_\Lambda)$  is a DVR whose residue field  $\kappa$  is a finite field or a characteristic 0 field. Let there be specified a parameter datum over  $S$ . Denote by  $X_{\overline{M}} \rightarrow \overline{M}$  a projective morphism (not necessarily flat) that restricts to  $X_M$  over  $M$ .

Assume that the parameter datum satisfies the RSC property. For every integer  $r \geq r_0$ , denote by  $C_{W_r} \subset W_r \times_{\operatorname{Spec} Q} M_Q$  the universal closed subscheme. Denote

by  $C_{G_r} \subset G_{\Lambda,r} \times_{\text{Spec } \Lambda} \overline{M}$  the closure of  $C_{W_r}$ . Denote the restrictions to  $C_{W_r}$  of the two projections as follows,

$$\rho_{W_r} : C_{W_r} \rightarrow W_r,$$

$$\rho_{M,r} : C_{W_r} \rightarrow M_Q.$$

By hypothesis,  $\rho_{W_r} : C_{W_r} \rightarrow W_r$  is smooth and projective with geometric fibers that are irreducible curves. For every integer  $e$ , denote by  $\text{Pic}_{C/W_r}^e \rightarrow W_r$  the relative Picard scheme of  $\rho_{W_r}$  parameterizing invertible sheaves of degree  $d$  on these curves. The relative degree over  $W_r$  of the pullback to  $C_{W_r}$  of  $\mathcal{O}_{\overline{M}_Q}(1)$  equals  $e(r) = e_0 r^{m-1}$  for an integer  $e_0$ . For every integer  $\ell$ , the pullback of  $\mathcal{O}_{\overline{M}_Q}(\ell)$  defines a global section of  $\text{Pic}_{C/W_r}^{\ell e(r)}$  over  $W_r$ .

Denote by  $f_r : X_r \rightarrow C_{W_r}$  the pullback by  $\rho_{M,r}$  of the universal family  $f_{M_Q} : X_{M_Q} \rightarrow M_Q$ . Denote by  $\overline{f}_r : \overline{X}_r \rightarrow C_{G_r}$  the closure of  $X_r$  in  $C_{G_r} \times_{\overline{M}} X_{\overline{M}}$ . Denote by  $\mathcal{L}_r$  the pullback of  $\mathcal{L}_Q$  to  $X_r$ . Consider the base change by the generic point  $\text{Spec } (Q(W_r)) \rightarrow W_r$  of  $X_r$ ,  $C_{W_r}$  and  $\mathcal{L}_r$ . By the RSC hypothesis, these base changes form a rationally simply connected fibration. Thus, by [dJHS11, Theorem 13.2], there exists a sequence  $(Z_{e,Q(W_r)})_{e \geq \epsilon'(r)}$  of irreducible components of the relative Hilbert scheme  $\text{Hilb}_{X_r/W_r}^{et+1-g_r} \times_{W_r, \text{Spec } Q(W_r)} \text{Spec } Q(W_r)$  satisfying the conditions listed above. For every  $e \geq \epsilon'(r)$ , define  $Z_e$  to be the closure of  $Z_{e,Q(W_r)}$  in the relative Hilbert scheme  $\text{Hilb}_{\overline{X}_r/G_r}^{et+1-g(C_Q)}$ . Define  $\epsilon(r)$  to be the smallest multiple  $\ell e(r)$  of  $e(r)$  such that  $\epsilon(r) \geq \epsilon'(r)$ .

**Proposition 4.5.** [SX, Proposition 1.18] *For every integer  $r > 0$  there exists an integer  $\epsilon(r) > 0$  with the following property. For every closed point of  $S$  with residue field  $\kappa$ , for every field extension  $L/\kappa$  such that  $L$  is RC solving, for every integral curve  $C_L \subset \overline{M}_L$  intersecting  $M_L$ , if  $\deg_{C_L}(\mathcal{O}_{\overline{M}}(1)) \leq r$  then there exists a curve  $\mathcal{C}_L \subset f_{\overline{M}}^{-1}(C_L)$  with  $f : \mathcal{C}_L \rightarrow C_L$  an isomorphism over the generic point and with  $\deg_{C_L}(\mathcal{L}_{\overline{M}}) \leq \epsilon$ .*

*Proof.* Having specified  $\epsilon(r)$  independent of  $\kappa$ , we are now free to prove the proposition after replacing  $S$  by  $\text{Spec } \Lambda$ . Since  $Q(W_r)$  has characteristic 0, by resolution of singularities, there exists a projective morphism  $\tilde{Z}_e \rightarrow Z_e$  such that the geometric generic fiber of  $\tilde{Z}_e \rightarrow G_{\Lambda,r}$  is smooth. There is an Abel map,

$$\tilde{\alpha}_e : \tilde{Z}_e \times_{G_r} W_r \rightarrow \text{Pic}_{C/W_r}^e.$$

Denote by  $U_e \subset \text{Pic}_{C/W_r}^e$  the maximal open subscheme over which  $\tilde{\alpha}_e$  is smooth and such that the fiber of  $\tilde{\alpha}_e$  over every geometric point of  $U_e$  intersects the dense open subset of  $\tilde{Z}_e$  parameterizing closed subschemes that are images of sections of  $f_r$ . In particular, for  $e$  equal to  $\epsilon(r)$ , define  $W_r^o \subset W_r$  to be the image of the dense open  $U_e$  under the smooth morphism  $\text{Pic}_{C/W_r}^e \rightarrow W_r$ .

By [SX, Corollary 3.15], the curve  $C_L$  is an irreducible component of multiplicity 1 in a curve in  $\overline{M}_L$  parameterized by a  $\Lambda$ -morphism  $z : \text{Spec } L \rightarrow G_{\kappa,r}$ . By Lemma 3.7 applied to the parameter space  $G_{\Lambda,r} \rightarrow \text{Spec } \Lambda$  and the dense open subscheme  $W_r^o$  of the generic fiber  $G_{Q,r}$ , for  $d = 0$  and  $E = F = L$ , there exists an integral extension of  $z$ , say

$$((B \rightarrow \text{Spec } \Lambda, z_B : B \rightarrow G_r), \psi_B : \text{Spec } L \rightarrow B_0).$$

Since  $\Lambda$  is Henselian,  $B$  has geometrically integral closed fiber and generic fiber over  $\text{Spec } \Lambda$ .

Define  $R$  to be the Henselization of the DVR  $\mathcal{O}$  obtained as the stalk of the structure sheaf of  $B$  at the generic point of  $B_0$ . Then  $\Lambda \rightarrow R$  is a regular local homomorphism,  $z_B$  induces a  $\Lambda$ -morphism,  $z_R : \text{Spec } R \rightarrow G_R$  mapping the generic point into  $W_r^o$ , and there exists a  $\Lambda$ -morphism  $\psi_R : \text{Spec } L \rightarrow \text{Spec } R$  whose composition with  $z_R$  equals  $z$ . By construction, for  $e = \epsilon(r) = \ell e(r)$ , the relative Picard  $\text{Pic}_{\mathbb{C}/W_r}^e$  has a section over  $W_r$  coming from  $\mathcal{O}_{\overline{M}}(\ell)$ . Thus the pullback by  $z_R$  has a section over  $\text{Spec } \text{Frac}(R)$ . Since  $R$  is Henselian, and since relative Picard scheme is smooth, there exists a section  $[\mathcal{A}]$  over  $R$  that maps into the open subscheme  $U_e$ .

Define  $\tilde{Z}_{R,\mathcal{A}}$  to be the closure in  $\tilde{Z}_e \times_{G_r} \text{Spec } R$  of the fiber of  $\tilde{\alpha}_e$  over  $[\mathcal{A}]$ . Then  $\tilde{Z}_{R,\mathcal{A}} \rightarrow \text{Spec } R$  is projective and flat. By construction, the geometric generic fiber is separably rationally connected. Since  $R$  is prime regular, and since  $L$  is RC solving, there exists an  $L$ -point of  $\tilde{Z}_{R,\mathcal{A}} \times_{\text{Spec } R} \text{Spec } L$ . By [SX, Proposition 4.1], the image of this  $L$ -point in the Hilbert scheme  $\text{Hilb}_{\overline{X}_r/G_r}^{et+1-g(C_Q)} \times_{G_r} \text{Spec } L$  parameterizes a closed subscheme of  $\overline{X}_r \times_{G_r} \text{Spec } L$  whose restriction over a dense open  $C_L^o$  in  $C_L$  is the image of a rational section of  $X_M \times_M C_L^o \rightarrow C_L^o$ . By construction, the total degree of this closed subscheme has degree  $e(r)$ . Thus, the closed image of the rational section has degree  $\leq e(r)$ .  $\square$

## 5. PERFECT PAC FIELDS AND GLOBAL FUNCTION FIELDS

Let  $S$  be  $\text{Spec } \Lambda$ , where  $\Lambda$  is a Henselian DVR with finite residue field  $\kappa$ . Fix a parameter datum over  $S$  with a codimension  $> 1$  compactification. A *global function field over  $M$*  is a function field over  $M$ ,

$$(\text{Spec } (R) \rightarrow S, E/F, z : \text{Spec } E \rightarrow M),$$

such that the residue field extension  $\kappa \rightarrow F = R/\mathfrak{m}_R$  is finite, i.e.,  $F$  is a finite field. Thus, the field  $E$  – a function field of a geometrically integral  $F$ -curve – is a global function field.

**Definition 5.1.** The parameter datum has *rational points over global function fields* if for every global function field over  $M$  as above, the base change  $X_E = \text{Spec } E \times_M X_E$  has an  $E$ -point.

**Proposition 5.2.** *If a parameter datum has rational points over global function fields, then for every irreducible closed subset of the closed fiber  $B \subset M_0$  with pullback family  $X_B = B \times_M X_M$ , there exists a PAC section of  $X_B \rightarrow B$ . Thus, for every field extension  $\kappa \rightarrow L$ , for every  $S$ -morphism  $\text{Spec } L \rightarrow M_0$ , the pullback family  $X_L = \text{Spec } L \times_M X_M$  has a PAC section over  $\text{Spec } L$ . In particular, if  $L$  is a perfect PAC field, then  $X_L$  has an  $L$ -point.*

*Proof.* First consider the case that  $B$  has dimension 0, i.e., as a  $\kappa$ -scheme this equals  $\text{Spec } F$  for a finite field extension  $F/\kappa$ . Define  $E$  to be  $F(t)$ , the function field of  $\mathbb{P}_F^1$ . Define  $z : \text{Spec } E \rightarrow M$  to be the composition of  $\text{Spec } E \rightarrow \text{Spec } F$  and the specified closed point  $\text{Spec } F \rightarrow M$ . Since the parameter datum has rational points over global function fields, there exists a lifting of  $z$  to an  $F$ -morphism  $s : E \rightarrow X_B$ . The closure  $Y$  of the image of this morphism is the image of an  $F$ -morphism from

$\mathbb{P}_F^1$ , and hence  $Y$  is geometrically irreducible over  $\text{Spec } F$ . Thus  $Y$  is a PAC section of  $X_B \rightarrow B$ .

Thus, without loss of generality, assume that  $B$  has dimension  $m \geq 1$ . Define  $F$  to be the algebraic closure of  $\kappa$  in the fraction field of  $B$ . Thus  $B$  is a geometrically integral  $F$ -scheme. Let  $u : B \rightarrow \mathbb{P}_F^N$  be a generically unramified, finite type morphism. Set  $c$  equal to  $m - 1$ , and set  $r$  equal to  $N - c$ . Thus,  $\text{Grass}_F(\mathbb{P}^r, \mathbb{P}^N)$  parameterizes linear subspaces of codimension  $c = m - 1$ , so that the inverse image under  $u$  has dimension  $\geq 1$ .

By way of contradiction, assume that  $f_B : X_B \rightarrow B$  has no PAC section. Then by Corollary 2.4, there exists a finite Galois extension  $F'/F$  and a dense open subset  $U \subset \text{Grass}_F(\mathbb{P}^r, \mathbb{P}^N)$  such that for every field extension  $K/F$  that is linearly disjoint from  $F'/F$ , for every  $[L] \in U(\text{Spec } K)$ , the curve  $C = B \times_{\mathbb{P}_F^N} L$  is a geometrically integral curve over  $K$  and for  $X_C := X_B \times_B C$ , the projection morphism  $f_C : X_C \rightarrow C$  has no PAC section. By elementary considerations of the intersection of  $U$  with any of the open affine spaces in  $\text{Grass}_F(\mathbb{P}^r, \mathbb{P}^N)$  forming an open Bruhat cell, or by the more refined Lang-Weil estimates, there exists an integer  $d_0$  such that for every finite field extension  $K/F$  of degree  $\geq d_0$ ,  $U(K)$  is nonempty. In particular, there exists such an extension of the finite field  $F$  of degree that is prime to  $d = [F' : F]$ , e.g., of degree  $dd_0 + 1$ . Since  $X_C \rightarrow C$  has no PAC section, it also has no rational section. This contradicts the hypothesis that the parameter datum has rational points over all global fields. This contradiction implies that  $X_B \rightarrow B$  does have a PAC section.

Now for a field extension  $\kappa \rightarrow L$  and a  $S$ -morphism  $z : \text{Spec } L \rightarrow M_0$ , define  $B \subset M_0$  to be the closure of the image of  $z$ . Then  $B$  is an integral closed subscheme of  $M_0$ . By the argument above,  $X_B \rightarrow B$  has a PAC section. The base change of this PAC section by the dominant morphism  $\text{Spec } L \rightarrow B$  is a PAC section of  $X_L \rightarrow \text{Spec } L$ .  $\square$

By [SX, Proposition 1.12], if the parameter datum satisfies the RSC property, then the parameter datum has rational points over global function fields.

**Corollary 5.3.** *For every parameter datum that satisfies the RSC property, the parameter datum has rational points over global function fields. Thus, for every extension field  $\kappa \rightarrow L$  for which  $L$  is a perfect PAC field, for every  $S$ -morphism  $z : \text{Spec } L \rightarrow M$ ,  $X_L$  has an  $L$ -point.*

## 6. THE GROTHENDIECK-SERRE CONJECTURE AND SERRE'S "CONJECTURE II" IN POSITIVE CHARACTERISTIC

The proof of Serre's "Conjecture II" for the function field  $k(S)$  of a surface over an algebraically closed field  $k$  of arbitrary characteristic was completed in [dJHS11, Theorem 1.5]. Since this is crucial for establishing Serre's "Conjecture II" for function fields of curves over perfect PAC fields that are nice, we briefly recall the proof and clarify one point.

The main step there is the proof of Serre's "Conjecture II" for those semisimple, connected, and simply connected groups over  $k(S)$  that are of the form  $G_0 \times_{\text{Spec } k} \text{Spec } k(S)$ , with  $G_0$  a semisimple, connected, and simply connected group over  $k$ . Such a group is necessarily split, since  $k$  is algebraically closed. In particular, since

the automorphism group of the split group  $G_0$  of type  $E_8$  is precisely  $G_0$  acting by conjugation, it follows that every group  $G$  over  $k(S)$  of type  $E_8$  is isomorphic to  $G_0 \times_{\mathrm{Spec} k} \mathrm{Spec} k(S)$ . Thus, the split case of Serre’s “Conjecture II” implies the  $E_8$  case of Serre’s “Conjecture II”. Combined with tremendous earlier work on Serre’s “Conjecture II”, the  $E_8$  case of Serre’s “Conjecture II” settles the full case of Serre’s “Conjecture II” over  $k(S)$ , when  $k$  has *characteristic 0*, cf. [CTGP04, Theorem 1.2(v)].

In fact, the proof in characteristic 0 implies the proof in arbitrary characteristic via the type of lifting results from [SdJ10] and further explored in [SX] and this article.

**Theorem 6.1.** *The characteristic 0 case of Serre’s “Conjecture II” for function fields of surfaces implies the characteristic  $p$  version. Precisely, for every algebraically closed field  $F$  of characteristic  $p$ , for every function field  $E/F$  of a geometrically integral  $F$ -scheme of dimension 2, for every semisimple algebraic group  $G_E$  over  $\mathrm{Spec} E$  that is connected and simply connected, for every torsor  $\mathcal{T}_E$  over  $\mathrm{Spec} E$  for  $G_E$ , there exists an  $E$ -point of  $\mathcal{T}_E$ .*

*Proof.* Let  $(\Lambda, \mathfrak{m}_\Lambda)$  be a Henselian DVR whose residue field  $\kappa$  is a finite field of characteristic  $p$  and whose fraction field  $Q$  has characteristic 0. Let  $H_0$  be a semisimple group that is connected and split. Let  $\kappa \rightarrow F$  be a field extension (not yet assumed algebraically closed), let  $E/F$  be the function field of a geometrically integral  $F$ -scheme of dimension  $d$  (not yet assumed equal to 2), let  $G_E$  be a linear algebraic group over  $E$  such that  $G_E \times_{\mathrm{Spec} E} \mathrm{Spec} \overline{E}$  is isomorphic to  $H_0 \times_{\mathrm{Spec} \kappa} \mathrm{Spec} \overline{E}$  as a group scheme over  $\overline{E}$ , and let  $\mathcal{T}_E$  be a  $G_E$ -torsor over  $\mathrm{Spec} E$ . By the proof of [SX, Corollary 1.22] and Lemma 3.7, there exists an integral model,

$$(B \rightarrow \mathrm{Spec} \Lambda, \pi_B^o : C_B^o \rightarrow B, (G_C \rightarrow C_B^o, \mathcal{T}_C \rightarrow C_B^o)),$$

and a pair

$$(\psi_B : \mathrm{Frac}(B_0) \rightarrow F, \psi_E : F(C_F^o) \rightarrow E)$$

as in Definition 3.5, and where  $G_C \rightarrow C_B^o$ , resp.  $\mathcal{T}_C \rightarrow C_B^o$ , is a semisimple group scheme, resp. is a torsor for this group scheme, such that the base change by  $\psi_E$  is  $(G_E, \mathcal{T}_E)$ . The choice of parameter datum  $M$  for this integral extension involves the automorphism group scheme of  $H_0$  and is fully explored in [SX, Proof of Corollary 1.22].

Now assume that  $d$  equals 2. The function field  $Q(B)$  is characteristic 0, so by the characteristic 0 case of Serre’s “Conjecture II”, there exists a finite extension  $Q(B')/Q(B)$  such that after base change by this extension, the torsor is trivial. After replacing  $B$  by a dense Zariski open whose complement has codimension  $\geq 2$ , which thus has nonempty intersection with the closed fiber  $B_0$ , there exists a finite, flat morphism  $B' \rightarrow B$  such that  $B'$  is integral and the associated extension of fraction fields is the extension  $Q(B')/Q(B)$  above. It may well happen that  $B' \times_{\mathrm{Spec} \Lambda} \mathrm{Spec} \kappa$  is reducible. Choose one irreducible component  $B'_0$ , and replace  $B'$  by the open complement of the remaining irreducible components. Then the residue field  $\kappa(B'_0)$  is a finite algebraic extension of  $\kappa(B_0)$ .

Assume now that  $F$  is algebraically closed. Then there is a factorization of  $\psi_B : \mathrm{Spec} E \rightarrow \mathrm{Spec} \kappa(B_0)$  through  $\psi_{B'} : \mathrm{Spec} E \rightarrow \mathrm{Spec} \kappa(B'_0)$ . Thus, up to replacing  $B$  by  $B'$ , assume that the generic fiber of  $\mathcal{T}_C \rightarrow C_B^o$  is a trivial torsor over the fraction field  $Q(C_B^o)$ . Denote by  $\mathcal{O}$  the DVR that is the stalk of the structure

sheaf of  $C_B^o$  at the generic point of the closed fiber  $C_0^o = C_B^o \times_{\text{Spec } \Lambda} \text{Spec } \kappa$ . Then the pullback of  $G_C$  and  $\mathcal{T}_C$  over  $\text{Spec } \mathcal{O}$  form a semisimple group scheme and a torsor for that group scheme over a DVR. By construction, this torsor is trivial when restricted to the fraction field  $Q(C_B^o)$  of  $\mathcal{O}$ . Thus, by Nisnevich's solution of the Grothendieck-Serre Conjecture over DVRs, [Nis84], the restriction of the torsor over the residue field of  $\mathcal{O}$  is also trivial. Taking the further base change by  $\psi_E$ , it follows that the original torsor  $\mathcal{T}_E$  over  $\text{Spec } E$  is trivial.  $\square$

## 7. PROOFS OF THEOREMS 1.1, 1.2, 1.3

By Theorem 3.13 and Theorem 3.14, every perfect PAC field  $L$  that is nice is RC solving. Thus, by Proposition 4.4, for every regular, integral Noetherian scheme  $S$  of dimension  $\leq 1$  whose function field has characteristic 0, for every parameter datum over  $S$  with a codimension  $> 1$  compactification that satisfies the RSC property, for every function field  $E = L(\eta)$  of a geometrically integral  $L$ -curve, for every  $S$ -morphism  $z_E : \text{Spec } E \rightarrow M$ , the pullback  $X_E$  of the universal family  $X_M \rightarrow M$  by  $z_E$  has an  $E$ -rational point.

**The  $C_2$  Property and the Period-Index Theorem.** By [Sta13, Proposition 1.19], there is a parameter datum as above whose universal family is the family of 2-Fano complete intersections in projective space, resp. the family of minimal homogeneous varieties. Thus, every function field  $L(\eta)$  of a geometrically integral curve over a perfect PAC field that is nice is  $C_2$ . One example of minimal homogeneous spaces are generalized Severi-Brauer varieties, i.e., a smooth projective  $E$ -scheme  $X_E$  such that  $X_E \otimes_{\text{Spec } E} \text{Spec } \overline{E}$  is isomorphic to a (standard  $A_n$ -type) Grassmannian  $\text{Grass}_{\overline{E}}(r, \overline{E}^n)$  whose associated Isom torsor reduces to the group of inner automorphisms (there are outer automorphisms only if  $n$  equals  $2r$ ,  $r > 1$ ). Thus, each generalized Severi-Brauer variety  $X_E$  has an  $E$ -point if (and only if) it has vanishing elementary obstruction, i.e., if there exists an invertible sheaf  $\mathcal{L}_E$  on  $X_E$  whose base change to  $X_E \times_{\text{Spec } E} \text{Spec } \overline{E}$  generates the Picard group. As explained in [Sta10, Theorem 11.1] (based on joint work with de Jong from [SdJ10]), this implies that Period equals Index for Severi-Brauer varieties over  $E$ .

**The Split Case of Serre's "Conjecture II". Full Serre's "Conjecture II" in Characteristic 0.** Next, as explained in the proof of [dJHS11, Theorem 1.4], existence of  $E$ -rational points on minimal homogeneous varieties implies Serre's "Conjecture II" for torsors over  $E$  for semisimple groups that are connected, simply connected, and *split*. As explained in the proof of Theorem 6.1, when  $E$  is of characteristic 0, this implies the full Serre's "Conjecture II". Thus, it only remains to prove Serre's "Conjecture II" in case  $L$  has characteristic  $p$ . Since  $L$  is nice,  $L$  contains the algebraic closure  $\overline{\kappa}$  of the prime field.

**Full Serre's "Conjecture II" in Positive Characteristic.** By the proof of Theorem 6.1, for every semisimple algebraic group  $G_E$  over  $E$  that is connected and simply connected, and for every  $G_E$ -torsor  $\mathcal{T}_E$ , there exists an integral model. In particular, the closed fiber of this integral model gives a smooth, quasi-projective  $\kappa$ -scheme  $B_0$  that is integral (but typically not geometrically irreducible), a quasi-projective, smooth morphism  $C_0^o \rightarrow B_0$  whose geometric fibers are irreducible curves, a semisimple group scheme  $G_{C,0} \rightarrow C_0^o$  whose geometric fibers are connected and simply connected, and a torsor  $\mathcal{T}_{C,0} \rightarrow C_0^o$  under  $G_{C,0}$ . Moreover, there is a homomorphism of  $\kappa$ -extensions  $\psi_B : \kappa(B_0) \rightarrow L$  and an associated isomorphism

of  $F$ -extensions  $\psi_E : F(C_F^o) \rightarrow E$  such that the base changes by  $\psi_E$  of  $G_{C,0}$ , resp.  $\mathcal{T}_{C,0}$  equal  $G_E$ , resp.  $\mathcal{T}_E$ .

Since  $B_0$  is a quasi-projective, smooth, irreducible  $\kappa$ -scheme,  $\kappa(B)/\kappa$  is a finitely generated field extension. Thus the algebraic closure  $\kappa'$  of  $\kappa$  in  $\kappa(B)$  is a finite extension of  $\kappa$ , and so it is again a finite field. Note that  $B$  is geometrically integral over  $\kappa'$ . Via the morphisms to  $B_0$ , the schemes  $C_0^o$ ,  $G_{C,0}$  and  $\mathcal{T}_{C,0}$  are all quasi-projective  $\kappa'$ -schemes.

Denote by  $C_0$  a projective compactification of  $C_0^o$ . Up to replacing  $B_0$  by a dense open subscheme, assume that  $C_0 \rightarrow B_0$  is projective and flat. Also, up to normalizing (all schemes are of finite type over  $\kappa'$ , so normalization is finite), assume that  $C_0^o$  is normal. Then the non-regular locus has codimension 2. Since  $C_0$  has relative dimension 1 over  $B_0$ , the image of the non-regular locus in  $B_0$  is a closed subset of codimension  $\geq 1$ . Up to shrinking  $B_0$  once more, assume that  $C_0$  is regular.

Denote by  $\overline{\mathcal{T}}_{C,0}$  a projective compactification of  $\mathcal{T}_{C,0}$ . As above, the non-flat locus of  $\overline{\mathcal{T}}_{C,0} \rightarrow C_0$  has codimension  $\geq 2$  in  $C_0$ . The image of this subset in  $B_0$  is a closed subset of codimension  $\geq 1$ . Thus, up to shrinking  $B_0$  once more, assume that  $\overline{\mathcal{T}}_{C,0} \rightarrow C_0$  is projective and flat.

Inside the relative Hilbert scheme  $\text{Hilb}_{\overline{\mathcal{T}}_{C,0}/B_0}$ , there is a locally closed subscheme  $\text{Sec}$  such that for every  $B$ -scheme  $T$ , the  $T$ -points of  $\text{Sec}$  are precisely those closed subschemes  $\overline{Z} \subset T \times_{B_0} \overline{\mathcal{T}}_{C,0}$  satisfying the following conditions: for the intersection  $Z$  of  $\overline{Z}$  with the open subscheme  $T \times_{B_0} \mathcal{T}_{C,0}$ , for the projection  $Z \rightarrow T \times_{B_0} C_0$ , the maximal open subset of  $T \times_{B_0} C_0$  over which this projection is an isomorphism is an open subset that surjects to  $T$ . Denote by  $Z^o$  the inverse image in  $Z$  of this open subset, so that  $Z^o \rightarrow T \times_{B_0} C_0$  is an open immersion. Thus,  $Z^o$  defines a rational section of the base change morphism,

$$\mathcal{T}_{C,0} \times_{B_0} \text{Sec} \rightarrow C_0 \times_{B_0} \text{Sec}.$$

Of course the scheme  $\text{Sec}$  has countably many connected components  $(\text{Sec}_i)_{i \in I}$  indexed by the countably many possible Hilbert polynomials of  $Z$ .

The claim is that there exists  $i \in I$  such that the morphism

$$\phi_i : \text{Sec}_i \times_{\text{Spec } \kappa'} \text{Spec } \overline{\kappa}' \rightarrow B_0 \times_{\text{Spec } \kappa'} \text{Spec } \overline{\kappa}',$$

has a PAC section. Assuming the claim, since  $L$  is assumed to contain  $\overline{\kappa}'$ , the base change  $\text{Spec } L \times_{B_0} \text{Sec}$  has an  $L$ -point. For this  $L$ -point, the rational section  $Z^o$  defines an  $E$ -point of  $\mathcal{T}_E$ . Thus, it suffices to prove the claim.

If  $B_0$  has dimension 0, then the claim is true. Indeed,  $B_0 \rightarrow \text{Spec } \kappa'$  is an isomorphism, and  $C_0 \rightarrow B_0$  is a geometrically integral curve over the finite field  $\kappa'$ . By Steinberg's Theorem, [Ste65], or by [dJS03], after base change to  $\text{Spec } \overline{\kappa}'$ , the pullback of the torsor  $\mathcal{T}_{C,0}$  over this curve has a rational section. Thus, assume that  $B$  has dimension  $m \geq 1$ .

The existence or non-existence of a PAC section of  $\phi_i$  is preserved by base change from the algebraically closed field  $\overline{\kappa}'$  by any extension  $\overline{\kappa}' \hookrightarrow k$  with  $k$  an algebraically closed field. Let  $u : B_0 \rightarrow \mathbb{P}_{\kappa'}^N$  be a generically unramified, finite type morphism. Define  $c$  to be  $m - 1$ , and define  $r$  to be  $N - c$ . Define  $k$  to be the algebraic closure of the function field of  $\text{Grass}_{\kappa'}(\mathbb{P}^r, \mathbb{P}^N)$ . Define  $[H] \in \text{Grass}_{\kappa'}(\mathbb{P}^r, \mathbb{P}^N)(\text{Spec } k)$  to be the  $k$ -point corresponding to the universal linear space. By the proof of



Theorem 2.2, the corresponding curve  $B_H = B \times_{\mathbb{P}_{\kappa'}^N} H$  is a smooth, irreducible, quasi-projective curve over  $k$ .

The base change of  $C_0 \rightarrow B_0$  by  $B_H \rightarrow B_0$  gives a projective, flat, generically smooth morphism  $C_H \rightarrow B_H$  with geometrically irreducible fibers. Since  $B_H$  is itself a smooth, irreducible, quasi-projective curve over the algebraically closed field  $k$ ,  $C_H$  is a generically smooth, irreducible, quasi-projective surface over  $k$ . The pullback of  $\mathcal{T}_{C,0}$  over the surface  $C_H$  is a torsor for a semisimple, connected, and simply connected algebraic group over  $C_H$ . By Theorem 6.1, there exists a rational section of this torsor. The closure of this section in the pullback of  $\overline{\mathcal{T}}_{C,0}$  defines a closed subscheme  $Z$ . The projection  $Z \rightarrow B_H$  is flat, since  $B_H$  is a smooth curve. Over a dense open subset of  $B_H$ , this closed subscheme defines a section of the restriction of  $\phi_i$ , for  $i$  equal to the Hilbert polynomial of the fibers of  $Z \rightarrow B_H$ .

Note that  $\overline{\kappa}'$  is already algebraically closed, so every finite Galois extension is an isomorphism, thus automatically linearly disjoint from every field extension  $K/k$ . For this choice of  $i$ , if there is no PAC section of  $\phi_i$ , then by Corollary 2.4, there exists a dense, Zariski open subset  $U_i \subset \text{Grass}_{\overline{\kappa}'}(\mathbb{P}^r, \mathbb{P}^N)$  such that for every field extension  $K/\kappa$  and for every  $[H'] \in U_i(\text{Spec } K)$ , there is no PAC section of the restriction of  $\phi_i$  over the curve  $B_0 \times_{\mathbb{P}_{\kappa'}^N} H'$ . For  $K$  equal to  $k$  and for  $[H]$ , there is a PAC section of the restriction of  $\phi_i$  over  $B_0 \times_{\mathbb{P}_{\kappa}^N} H$ . Therefore, there does exist a PAC section of  $\phi_i$ . Thus, the torsor  $\mathcal{T}_E$  has an  $E$ -point.

**Low Degree Complete Intersections in Grassmannians.** By [SX, Proposition 1.19], there exists a parameter datum  $M$  with a codimension  $> 1$  compactification for pairs  $(X, \mathcal{L})$  of polarized schemes whose base change to an algebraic closure is isomorphic to  $\text{Grass}(r, K^{\oplus n})$  with its Plücker invertible sheaf. Denote by  $P \rightarrow M$  the projective bundle parameterizing 1-dimensional subspaces of  $H^0(X, \mathcal{L}^{\otimes d})$ . This admits a codimension  $> 1$  compactification  $P \hookrightarrow \overline{P}$  by the same GIT construction used to construct the codimension  $> 1$  compactification of  $M$ . Inside  $X_M \times_M P$ , define  $Y_P$  to be the closed subscheme that is the zero scheme of the corresponding global section of  $\mathcal{L}^{\otimes d}$ . The projection  $Y_P \rightarrow X_M$  is a projective bundle. Thus  $Y_P$  is smooth over  $S$ . Thus, by [SX, Lemma 4.4], the datum satisfies the first hypothesis of the RSC property. The inequality  $(3r - 1)d^2 - d < n - 4r - 1$  implies the inequality  $r(n - r) \geq 3$ . Thus, by [SX, Corollary 4.6], the datum satisfies the second hypothesis of the RSC property. The third hypothesis of the RSC property follows by construction: since  $\mathcal{L}$  is relatively ample for  $X_M \rightarrow M$ , it is also relatively ample for  $X_M \times_M P \rightarrow P$ , and thus its restriction to  $Y_P$  is relatively ample for  $Y_P \rightarrow P$ . Finally, hypotheses four, five, and six are verified in the PhD thesis of Robert Findley, [Fin10].

## 8. PROOF OF THEOREM 1.4

Let  $(R, \mathfrak{m}_R)$  be a Henselian DVR with residue field  $k$  (no hypothesis yet on  $k$ ) and with fraction field  $K$ . Assume that  $R$  is excellent, i.e.,  $\text{Frac}(\widehat{R})/K$  is a separable extension. This holds automatically if  $K$  has characteristic 0 or if  $R$  is complete.

**Lemma 8.1.** *If the residue field  $k$  is separably closed, then  $K$  has cohomological dimension  $\leq 1$ . If  $k$  is algebraically closed, then  $K$  has dimension  $\leq 1$ , and it is even  $C_1$ . If the cohomological dimension of  $k$  is  $\leq 1$ , then the cohomological dimension of  $K$  is  $\leq 2$  under either of the following conditions: if  $K$  has characteristic 0 or*

if  $R$  is complete. If  $k$  is perfect of dimension  $\leq 1$ , then for every finite extension  $K'/K$ , for every Severi-Brauer variety over  $K'$ , the Period equals the Index.

*Proof.* Let  $K'/K$  be any finite algebraic extension. Since  $R$  is excellent, the integral closure  $R'$  of  $R$  in  $K'$  is a finite  $R$ -module. Moreover,  $R'$  is a semilocal ring whose localization at any maximal ideal is a DVR. The residue fields of  $R'$  are finite extensions of  $k$ . The above hypotheses on  $k$  are each preserved under finite field extension. Thus, the localizations of  $R'$  satisfy the same (respective) hypotheses as  $R$ . Thus, every argument below for  $K$  also applies to  $K'$ .

If  $K$  is complete, and if  $k$  has cohomological dimension  $\leq 0$ , resp.  $\leq 1$ , then also  $K$  has cohomological dimension  $\leq 1$ , resp.  $\leq 2$ , [Ser02, Proposition II.12, p. 85].

Next consider the case that  $K$  is not complete. For every prime  $\ell$  different from the characteristic of  $K$ ,  $\text{cd}_\ell(K) \leq 1$  if and only if  $\text{Br}(K)[\ell] = \{0\}$ , [Ser02, Proposition II.4, p. 76]. Given a Severi-Brauer variety  $X_K \rightarrow \text{Spec } K$  whose period equals  $\ell$ , there is a projective, flat model  $X_R \rightarrow \text{Spec } R$  (typically ramified over the closed point). If  $\text{Frac}(\widehat{R})$  has cohomological dimension  $\leq 1$ , then  $X_R \times_{\text{Spec } R} \text{Spec } \widehat{R}$  has an  $\widehat{R}$ -point. Then by approximation [Gre66, Theorem 1], also  $X_R$  has an  $R$ -point. If  $K$  has characteristic  $p$ , then the  $p$ -cohomological dimension of  $K$  is  $\leq 1$ , [Ser02, Proposition II.3, p. 75]. Thus, if  $\text{Frac}(\widehat{R})$  has cohomological dimension  $\leq 1$ , then also  $K$  has cohomological dimension  $\leq 1$ . Please note: this argument does not say anything about  $\text{Br}(K)[p]$ .

By [Lan52], if  $R$  is complete and  $k$  is algebraically closed, then  $K$  is a  $C_1$ -field. Again applying [Gre66, Theorem 1], this also holds if  $R$  is excellent but not necessarily complete.

Next assume that  $K$  has characteristic 0 and that  $k$  has cohomological dimension  $\leq 1$ . Then  $\text{Frac}(\widehat{R})$  has cohomological dimension  $\leq 2$ . By the Merkurjev-Suslin Theorem, [Sus84, Corollary 24.9], the field  $K$ , resp.  $\text{Frac}(\widehat{R})$ , has cohomological dimension  $\leq 2$  if and only if, for every central simple algebra over  $K$ , resp. over  $\text{Frac}(\widehat{R})$ , the reduced norm is surjective. For a central simple algebra  $A_K$  over  $K$ , this extends to a finite, flat  $R$ -algebra  $A_R$  (possibly ramified at the closed point). For every  $r \in R$ , the equation  $\text{Nrd}_{A_R/R}(x) = r$  have a solution in  $\widehat{R}$ , since  $\text{Frac}(\widehat{R})$  has cohomological dimension  $\leq 2$ . Thus, again applying [Gre66, Theorem 1], also there is a solution over  $R$ . Thus  $K$  has cohomological dimension  $\leq 2$ .

Finally, assume that  $k$  is perfect. Since  $R$  is Henselian, for every  $n \geq 0$ , the pullback map

$$H_{\text{ét}}^n(\text{Spec } k, \mathbb{G}_m) \rightarrow H_{\text{ét}}^n(\text{Spec } R, \mathbb{G}_m)$$

is an isomorphism. By [Gro68, Proposition 2.1, p. 93], there is a restriction exact sequence

$$0 \longrightarrow \text{Br}(k) \longrightarrow \text{Br}(K) \longrightarrow H_{\text{ét}}^1(\text{Spec } k, \mathbb{Q}/\mathbb{Z}) \longrightarrow H_{\text{ét}}^3(\text{Spec } k, \mathbb{G}_m) \dots$$

If  $k$  has cohomological dimension  $\leq 1$ , this gives an isomorphism,

$$\mathbf{Br}(K) \xrightarrow{\cong} H_{\text{ét}}^1(\text{Spec } k, \mathbb{Q}/\mathbb{Z}).$$

Via the short exact sequence of Abelian groups,

$$0 \longrightarrow (1/d)\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{d} \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

the  $d$ -torsion in the Brauer group is identified with  $H_{\text{ét}}^1(\text{Spec } k, (1/d)\mathbb{Z}/\mathbb{Z})$ . Thus, for every element  $\alpha$  of order  $d$  in the Brauer group, the associated torsor over  $k$  gives a Galois field extension  $k'/k$  with cyclic Galois group of order  $d$ . There is an associated étale extension  $R \rightarrow R'$  of degree  $d$  that is also a local homomorphism of DVRs. The pullback of  $\alpha$  to  $\text{Spec } k'$  is the zero class. Thus, comparing exact sequences for  $R$  and  $R'$ , the pullback of  $\alpha$  to  $R'$  is the zero class. Thus, the index of  $\alpha$  equals  $d$ .  $\square$

Let  $S$  be a dense open subscheme of the spectrum of the ring of integers of a number field. For every parameter datum over  $S$ , by the extension Proposition 4.4 of [SX, Corollary 1.4], which in turn relies on [Den16, Section 7], there exists an integer  $p_0$  such that for every closed point  $\text{Spec } \kappa \rightarrow S$  of characteristic  $p \geq p_0$ , for every field extension  $\kappa \hookrightarrow L$  with  $L$  a perfect PAC field that is nice, for every pair of Henselian DVR over  $S$ ,  $\text{Spec } A \rightarrow S$  with  $\mathfrak{m}_A = \theta A$ , resp.  $\text{Spec } R \rightarrow S$  with  $\mathfrak{m}_R = \pi R$ , each having isomorphic residue field extensions of  $\kappa$ ,  $\bar{\tau} : A/\mathfrak{m}_A \xrightarrow{\cong} R/\mathfrak{m}_R$ , both equal to  $\kappa \hookrightarrow L$ , for every  $S$ -morphism  $z_A : \text{Spec } A \rightarrow M$ , the pullback  $z_A^* X_M$  has an  $A$ -point if and only if for every  $S$ -morphism  $z_R : \text{Spec } R \rightarrow M$ , the pullback  $z_R^* X_M$  has an  $R$ -point. This uses the following isomorphism of commutative monoids under multiplication  $\mathcal{MR}(A) = A/(1 + \mathfrak{m}_A)$ , resp.  $\mathcal{MR}(B) = B/(1 + \mathfrak{m}_B)$ , given by

$$\tau_{\theta, \pi} : \mathcal{MR}(A) \rightarrow \mathcal{MR}(R), \quad \tau_{\theta, \pi}(\text{mres}(u\theta^n)) = \text{mres}(v\pi^n),$$

where  $u \in A \setminus \mathfrak{A}$ , resp.  $v \in R \setminus \mathfrak{R}$ , are units such that  $\bar{\tau}(\bar{u})$  equals  $\bar{v}$  as elements in  $L$ .

Assume now that the parameter datum has a codimension  $> 1$  compactification and has the RSC property. By the extension of [SX, Corollary 1.13], for  $A = L[[t]]$ , for every morphism  $z_A : \text{Spec } L[[t]] \rightarrow M$ ,  $z_A^* X_M$  does have an  $A$ -point. Thus, for every Henselian DVR  $R$  over  $S$  with residue field extension  $\kappa \hookrightarrow L$ , for every  $S$ -morphism  $z_R : \text{Spec } R \rightarrow M$ , also  $z_R^* X_M$  has an  $R$ -point.

Applying this to the parameter datum from the previous section for complete intersections in projective space, resp. hypersurfaces in Grassmannians (both of which are defined over  $\text{Spec } \mathbb{Z}$ ), for every  $(n; d_1, \dots, d_c)$  with  $d_1^2 + \dots + d_c^2 < n-1$ , resp. for every  $(n, r, d)$  with  $(3r-1)d^2 - d < n-4r-1$ , there exists an integer  $p_0$  such that for every  $p \geq p_0$ , for every Henselian DVR  $R$  with residue field  $L$  a perfect PAC field of characteristic  $p$  that contains a primitive root of unity of order  $n$  for every integer  $n$  prime to  $p$ , there is a  $K$ -point of every  $(X_K, \mathcal{L}_K)$  over  $\text{Spec } K$  whose base change to  $\bar{K}$  is the common zero scheme in  $\mathbb{P}_{\bar{K}}^{n-1}$  of hypersurfaces of degrees  $(d_1, \dots, d_c)$  together with the restriction of  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ , resp. the base change is isomorphic to a degree  $d$  hypersurface in  $\text{Grass}_{\bar{K}}(r, \bar{K}^{\oplus n})$  together with its Plücker invertible sheaf.

Finally, applying this to the parameter datum for torsors for the split group of type  $E_8$ , there exists an integer  $p_0$  such that for every  $p \geq p_0$ , for every DVR as above with  $L$  of characteristic  $p \geq p_0$ , every torsor over  $\text{Spec } K$  for the split group of type  $E_8$  is trivial. Since the center of this group is trivial, and since there are no outer automorphisms for this group, also every torsor for the automorphism group of this group is a trivial torsor. Thus every form of  $E_8$  over  $\text{Spec } K$  is isomorphic to the split form. Therefore, every torsor for a group of type  $E_8$  over  $\text{Spec } K$  is a trivial torsor. By Lemma 8.1 also the characteristic 0 field  $K$  has cohomological

dimension 2 and satisfies Period equals Index. By [CTGP04, Theorem 1.2], Serre’s “Conjecture II” holds for  $K$ .

## 9. PROOF OF THEOREM 1.3

For 2-Fano complete intersections in projective space, resp. for low degree hypersurfaces in Grassmannians, there exists a parameter datum for these schemes over  $\text{Spec } \mathbb{Z}$  with a codimension  $> 1$  compactification, and the parameter datum satisfies the RSC property, cf. the previous section. Thus, by Corollary 5.3, every perfect PAC field of positive characteristic has rational points for these schemes.

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